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Deflection of rigid frames stressed beyond the yield point

Walter Thomas Daniels
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DEFLECTION OF RIGID FRAMES STRESSED BEYOND THE YIELD POINT

by

Walter Thomas Daniels

A Thesis Submitted to the Graduate Faculty
for the Degree of

DOCTOR OF PHILOSOPHY

Major Subject Theoretical and Applied Mechanics
and Structural Engineering

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Dean of Graduate College

Iowa State College
1941

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I. INTRODUCTION

A. Applicability of Rigid Frames in Construction

Frames made up of elements of any cross section and designed with stiff connections between the members at the joints, in order that loads may be supported, are called rigid frames. The connections must be capable of transferring axial tensile and compressive forces and bending moments. Further, the rigid frame is usually statically indeterminate; that is, the stresses in them can be found only by taking into account the relative stiffness and deformation of the members.

The rigidly connected frame has frequent application in steel and reinforced concrete construction and common use of the frame is made in engineering structures. Since about 1900 rigid frame construction has been used extensively in England, France, and Germany. Rigid frame construction was introduced in the United States in 1885 and at present use is made of this type of construction for buildings, subways, bridges, viaducts, culverts, water tanks, and reservoirs.

There is available sufficient literature for the analysis of a rigid frame loaded in any manner, when the material is stressed below the proportional limit. There exists comparatively little literature dealing with the plastic behavior of a rigid frame. Plastic behavior is distinguished from elastic behavior as the action of a frame when any part of the frame

is stressed above the proportional limit of the material. The scarcity of literature upon plastic behavior of a frame is perhaps due, partly, to the assumption that the members of the structure during its life do not experience a stress beyond the proportional limit. In consequence the structural members are designed by the theory of elasticity, and a factor of safety included in order to keep the stresses within the proportional limit.

B. Need for Study

It is sometimes of practical importance to know the results of stressing any part of a rigid frame above the proportional limit of the material, for the beginning of yielding may not mean failure of the structure. Moreover, in certain types of construction large plastic deflections of the structural members may be permissible, and the engineer should know the additional load the structure will carry. It is further assumed that the permanent strains in the members of the structure are also permissible. This is particularly true in the design of the fuselage of an airplane, a bent for a bomb shelter and buildings that are military objectives.

Notwithstanding the importance of the problem of plastic behavior no publications are available which deal with the determination of the deflections that occur in a rigid frame when the material is stressed above the proportional limit. There are available a few studies concerned with the plastic behavior of a beam in bending. There is available one publication (2) dealing with the moment determination in statically

indeterminate structures when plastic action exists in the structure.

This investigation is made in order to present a method that will enable the designer to predict with a reasonable degree of accuracy the deflections, moments, and angle changes in a particular type of rigid frame when a part of the frame is stressed beyond the yield point of the material. This type of frame finds common use in building, bridge, storage, and underground structures. It is thought, further, that the investigation may form a sound basis for analysis of other types of frames.

C. Review of Literature

Since bending of a member below the elastic limit necessarily precedes plastic bending, it is well to review briefly the flexure theory. According to Love (3) Galileo (1564-1642) is given credit for the first inquiry into the nature of the resistance of solids to rupture. He attempted to determine the resistance of a beam one end of which is built into a wall, when the tendency to break it arises from its own or an applied weight, and he concludes that the beam tends to turn about an axis perpendicular to its length, in the plane of the wall and at the bottom of the beam. Moore (5) adds that Galileo treated a solid as inextensible and incompressible until it broke. Moore (5) states, further, that Galileo was the first to state the principle of resisting and bending moment (although he did not use the same terms) very much in the same manner of the modern engineering textbooks.

Following the investigations of Galileo, Hooke and Mariotte (1620-1684) occupied themselves with the experimental discovery of what we now

term stress-strain relations. Hooke in 1678 gave the famous law of proportionality of stress and strain which bears his name, in the words, "Ut tensio sic vis; that the power of any spring is in the same proportion with the tension thereof." Mariotte according to Moore (5) is the next important figure in the development of the flexure formula. Mariotte made the assertion without any clear evidence, experimental or logical, that at mid-depth of a beam there is a layer which is neither stretched nor compressed, and this is the idea of the neutral axis. The idea of the neutral surface caused much discussion and was not settled until well into the eighteenth century when Coloumb's works were published.

James Bernoulli (1654-1705) in 1694 studied the differential equation of the elastic curve of the beam and shortly afterwards Parent presented the idea of the total compressive forces balancing the total tensile forces at any cross section of a beam. It is thought that Parent came very close to stating the modern flexure formula.

Navier (1785-1836) integrated all the previous works upon the formula and receives credit for the formula in its final form. However, he made the error, in his first treatment, of assuming the moments of the compressive forces always equal to the moments of the tensile forces. This is true only for symmetrical sections.

Finally, in 1864, B. de St. Venant (1797-1886) published what is now known as the complete mathematical basis for the theory of elasticity which included the principle of a plane section before bending remaining plane after bending. He reported explicitly the cases in which a plane section remains plane. His works are a classic for the mathematical

analysis of structural members elastically bent. He is also known for his works in torsion - both elastic and plastic.

St. Venant (8) is given credit for one of the first attempts to solve the problem of plastic bending. In fact, he indicated the relation between the bending moment, and the beam dimensions, radius of curvature of the beam, and a constant 'K' which depends upon the modulus of elasticity of the material and the part of the beam that remains elastic. The equation for the case of plastic bending produced by equal or 'circular' flexion of a beam of rectangular section is written:

$$M = 4 Kb(c^2 - \frac{4 K^2 p^2}{3 E^2}).$$

M is the bending moment, 2c represents the depth of beam, 2b represents the width of beam, 2K is the longitudinal stress in the plastic area, E represents Young's modulus of elasticity, and ρ is the radius of curvature.

In 1905 Bach and Ludwik (1) made investigations with a rectangular cast-iron beam in order to predict the rupture moment of the beam. Numerous test specimens for tensile and compressive tests were shaped out of the same material as the beam which was to be tested in flexure. A stress-strain diagram was drawn from the tensile and compressive test data. Use was made of the stress-strain diagram to compute the rupture moment. The development of equation 4 in this thesis shows clearly how the diagram may be utilized in order to determine the internal resisting moment. Bach and Ludwik were able to predict the rupture moment of a cast-iron beam, of cross section 8.01 cm by 8.006 cm and 1.0 m long, simply supported and loaded at its center with a concentrated load, to within 3.3 per cent. The

same investigators made other tests similar to this one with beams of circular cross section and made of materials other than cast iron, but the results of these data are not available.

In 1906 Herr W. von Pinegin (7), in the materials laboratory of Technischen Hochschule, Charlottenburg, made rupture tests in bending with five cast-iron beams - three rectangular and two of I cross section - in order to determine the rupture moments. The results were not as satisfactory as those of Bach and Ludwik. The computed moment - and it was determined in the same manner as Bach and Ludwik had determined it - was approximately 86 per cent of the observed moment. Since the beams were all tested simply supported with a concentrated load at the center, the observed moment is computed by multiplying the breaking load by one fourth the span. Herr W. von Pinegin gave no clear explanation of the reason for the disagreement of the moment values, but explicitly stated that he felt it was not an experimental error. However, Eugene von Meyer (4) disagreed with this statement and was of the opinion that von Pinegin's method of obtaining strains was, perhaps, unsatisfactory.

In 1908 (4) Professor Eugene von Meyer, of Charlottenburg, published the results of an inclusive set of tests with beams in which the material does not obey Hooke's Law. The experimental investigation was made with simply supported beams loaded at the center with a concentrated load.

The deflection was measured at the center and the angle change at the support was measured. Three cast-iron beams of circular cross section, one of square cross section, and two of rectangular cross section were tested. Likewise two mild steel beams of rectangular cross section were

tested. The cast-iron beams were tested with a span length of 1,000 mm and the mild steel beams were tested with a span length of 1,300 mm. Professor von Meyer developed from the same basic theory as Bach and Ludwik equations for the center deflection and angle change at the support. These equations may be solved graphically. Tabular data indicate that he obtained very close agreement between observed and computed values of deflections and angle changes. The values of the deflection, computed and observed, were within 5 per cent of one another. For the beams of mild steel the angle change values were indicated by curves and the observed and computed values agreed closely. It should be added that Professor von Meyer took into account the deflection due to shear. In some cases the shear deflection amounted to almost 9.3 per cent of the total deflection.

Dr. Hans Bleich (2) in 1932 developed a theory of plastic bending for beams which is the basis for the 'limit' design method as reported by Professor Van den Broek in 1939 (10). Dr. Bleich introduced the idea of the ideal stress-strain diagram for plastic bending and proceeded to solve for the ratio between the absolute bending moment that a section may develop and the maximum value it develops when acting purely elastically. He extended the theory to statically indeterminate structures and determined the moment curves for members of the structure.

In 1938, at the University of Michigan, Dr. E. O. Scott (9) made an investigation, which furnished data for a doctoral thesis, of four beams involving ductile behavior. Dr. Scott sought to determine the role that ductility plays in deformation and to what extent it should be included in specifications. A circular, rectangular, triangular, and 'I' section

beam were tested as cantilevers. All of these beams were made of mild steel. Dr. Scott writes: "There is a difference between capacity load and elastic load. Capacity load is defined as the increase in the useful limiting load for which a beam is acting in an entirely elastic manner." He concludes from the results of the investigation that the difference between capacity load and elastic load is sufficient to include ductility in specifications since the increase in the useful limit varies from 13.7 per cent for the 'I' section to 134 per cent for the triangular section. Dr. Scott measured the deflection at the center of all beams tested and determined the deviation of displacement for the beams between purely elastic and plastic behavior. Finally, to complete the investigation he made an analytical investigation of a beam with fixed ends, rectangular cross section, and carrying a uniform load. He concludes from the study of this beam that the deviation of the displacement from that which would obtain if the beam remained entirely elastic is as great as 167 per cent.

Unpublished at this time is an investigation of the plastic bending of beams by Mr. George Winter (11) at Cornell University, Ithaca, N. Y. Mr. Winter writes that he is seeking to develop a theory which will predict deflections for any load up to the ultimate load. First results of the tests, according to Mr. Winter, show that the ideal 'plastic hinge' can never be realized although for beams of ductile material it may be closely approached. Winter defines the ideal 'plastic hinge' as occurring when: "... the two parts of the beam at both sides of the cross section can be rotated with respect to each other through considerable angle without increasing the load, the internal resisting moment remaining un-

changed during this rotation." He states that the reason the ideal hinge can never be realized is that the material contains a small amount of work-hardening. This indicates the stress-distribution diagram is never a rectangle as assumed for the ideal case.

Dr. Nadai (6) has investigated the problem of plastic bending of beams in which strain hardening is taken into account. He has also made experimental investigations in order to determine the plastic boundary curve when mild steel beams of rectangular cross section are plastically bent. Dr. Nadai has also made an analytical investigation of ^{a beam of} the triangular cross section, simply supported and loaded at the center, when the beam is plastically bent. He develops, in a simple manner, the equation for the internal resisting moment of the beam, the equation for the plastic boundary curve, and shows how the neutral axis position may be determined for loads which cause the beam to be plastically bent.

D. The Problem

The review of literature indicates clearly the lack of any great amount of investigation with the rigid frame if plastic action exists in the frame. No literature is available which deals with the prediction of deflections of the members of the frame when a member is plastically bent. None of the investigations published have tested the validity of a prediction equation for the deflection of a beam that has a point load, in the center and equal moments on the ends of the beam, when that beam is plastically bent. In consequence this investigation does not duplicate

any former experimental investigations dealing with the plastic behavior of a beam or a rigid frame.

The prediction of the deflection of members of a rigid frame as a function of the applied load, the length and cross section of the members, and properties of the material, when plastic action exists in a rigid frame is, therefore, the aim of this investigation. The investigation includes a mathematical analysis and experimental tests. It deals specifically with 'U' type of frame, made of mild steel, and its members connected by welded joints. This type of frame is found desirable for use in hangars, underground tunnels, bomb shelters, and building structures for military uses. It is a type of frame of which common use is made in other structures such as viaducts, bridges, box culverts, and water tanks. The investigation includes a beam of the same material as the members of the frame in order to test the applicability and reliability of a mathematical analysis in predicting plastic deflections due to bending. Deflections are to be measured at several points along the beam member of the frame and compared with values obtained from a mathematical analysis.

Also, for the frame, the angle changes at the joints, frame spread and strains are to be measured and comparisons made with analytical values. For the beam tested with a point load at the center of the span and equal moments on the ends only deflections were measured and comparisons made with mathematically determined values of deflection.

II. THEORY AND FUNDAMENTAL EQUATIONS

A. Theory

If a test specimen of mild steel is subjected to an increasing load it is observed that the specimen will lengthen an amount proportional to the applied load. This proportionality between applied load and the increase in the length is constant until a certain stress is reached. This stress, defined as the yield point stress, is found by dividing the applied load by the cross section area of the specimen. If a definite length of the bar is measured the unit strain may be determined by dividing the measured length into the total extension. Let the stress found by dividing the applied load by the area of the specimen be called the unit stress, and designated by S'_0 . If the values of unit strain of the bar are plotted as abscissa and the unit stress as ordinates, in a right angled coordinate system, a curve such as shown in Figure 1 is obtained, up to the point A.

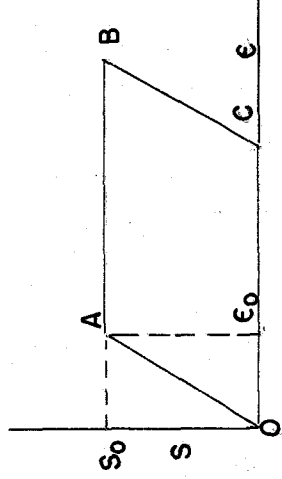


Fig. 1

If an attempt is made to increase the load, large strains are observed with very little increase of the load. This indicates that the

tensile stress beyond a value, designated S_0 , and called yield point stress, is not increased appreciably. It may, therefore, be assumed without appreciable error that the curve is a straight line parallel to the strain axis up to a point, say B.

If the specimen is unloaded after the curve reaches point B, the line BC will represent the unloading curve. The amount of permanent strain left in the test specimen for each unit of length is represented by OCC.

Such a stress-strain curve is designated an idealized one. It is idealized, for experimenters have found that near the point A the curve is not exactly straight and further that the material may show a value for the yield point slightly different from the one corresponding to S_0 . Some investigators, however, claim that the curve is exactly as drawn in Figure 2, and variations from the idealized curve may be due to the test specimen or the rate of running the machine.

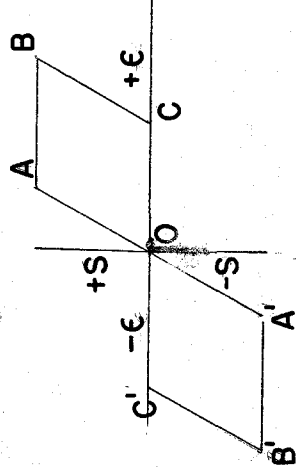


Fig. 2

Similar tests have been made in order to determine the stress-strain curve for mild steel in compression and the resulting curve is practically coincident with the tensile curve. Consequently the modulus of elasticity in tension may be assumed equal to the modulus in compression. If a negative sign is applied to compressive stress and compressive strains the diagram will appear as shown in Figure 4.

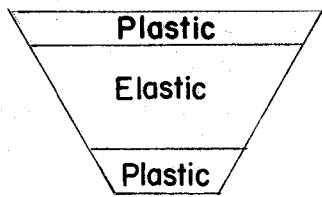


Fig. 3

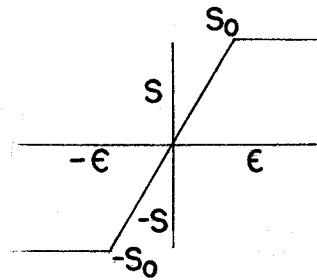


Fig. 4

When a bar of mild steel is subjected to bending by a load it is found that at a certain load the bar becomes plastically bent in some parts. At any section where the bar is plastically bent there are three distinct regions of strain. ^{see fig 3.} The center part will be strained elastically and above and below the elastic portion are parts that are strained plastically. For such a bar the idealized stress-strain diagram is shown by Figure 2. The plastic strains are assumed to be no larger than a few per cent of unity. In the center part of the beam cross section the strains are assumed not larger than the maximum unit elastic strain computed by dividing the unit stress by the Young's modulus.

If a bar of mild steel is bent plastically there are two kinds of stress regions inside the bar. At the top and bottom of the bar cross section are stresses which are equal to the yield point stress of the material. In the center of the section the stresses are elastic and are linearly proportional to the strains. A stress distribution diagram at the section is shown by Figure 5. If the yield stress in tension is assumed equal to the yield stress in compression and of a constant value, S_0 , the stress distribution diagram may be represented by three straight lines. In the

plastic regions of the section the stress s equals s_0 and within the elastic part of the section $s = E\epsilon$. E , is Young's modulus and ϵ is the unit strain which exists at the point where s is desired. It must be noted that the strains are assumed constant in a horizontal plane parallel to the neutral surface of the section.

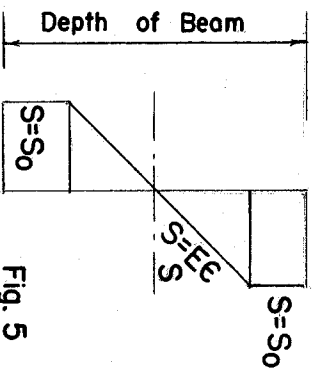


Fig. 5

B. Assumptions

In this investigation two sets of equations are to be developed.

One set applies in the case of purely elastic bending and the other applies to plastic bending. The assumptions in the case of purely elastic bending of a beam are all well known. For the plastically bent beam it may be assumed that the plane section before bending remains plane after bending and for the longitudinal fibers of the beam there exists during bending the same relation between stress and strain as in the case of simple tension and compression.

It is found from the investigations heretofore made that these two assumptions are valid. Further, it is found that deflection caused by shearing stresses may in some cases be neglected.

In this investigation the following assumptions are made in the

development of the equations for the plastic behavior of a frame member or a beam:

- (1) A PLANE SECTION, OF A STRUCTURAL MEMBER, REMAINS PLANE DURING BENDING BEYOND THE PROPORTIONAL LIMIT
- (2) HOOKE'S LAW IS VALID UP TO THE YIELD POINT OF THE MATERIAL
- (3) THE MATERIAL IS ISOTROPIC
- (4) SHEARING DEFLECTION IS NEGLIGIBLE
- (5) A STRUCTURAL MEMBER IS STRAIGHT AND OF UNIFORM CROSS SECTION
- (6) THE MODULI OF ELASTICITY IN TENSION AND COMPRESSION ARE EQUAL
- (7) THE IDEALIZED STRESS-STRAIN DIAGRAM FOR PURE BENDING REPRESENTS THE TRUE STRESS-STRAIN DIAGRAM FOR THE MATERIAL USED IN THIS INVESTIGATION
- (8) FORCES CAUSING DIRECT TENSION OR COMPRESSION ARE NEGLIGIBLE IN THEIR EFFECT UPON THE DEFORMATION OF A STRUCTURAL MEMBER

C. Equations for Internal Moment of Symmetrical Section

The stress variation across a symmetrical section corresponding to an idealized stress strain diagram may be represented as in Figure 5. The curve shows three straight lines, $s = E\epsilon$ for the elastic stresses, $s_o = E_t \epsilon_o$ for the plastic stresses in tension, $s_o = E_c \epsilon_o$ for the plastic stresses in compression. E_t and E_c represent the moduli of elasticity in tension and compression, s_o the yield point stress of the material, and ϵ_o the yield point strain.

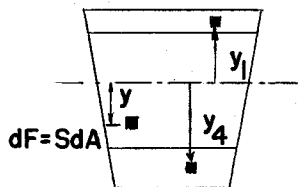
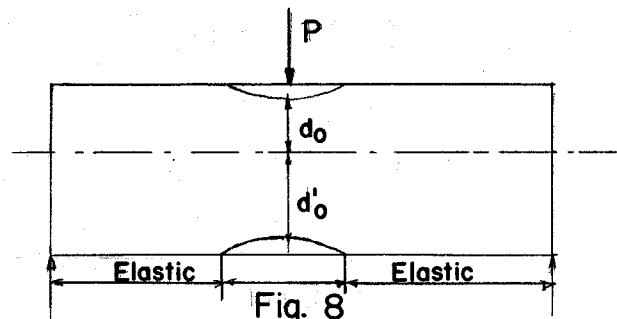


Fig. 6

Fig. 7
on page 21



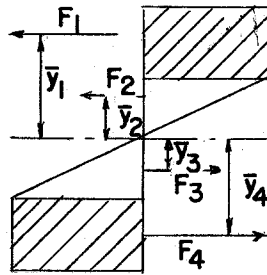


Fig. 7

Let F_1 and F_4 represent the forces due to purely plastic stresses, and F_2 and F_3 the forces due to purely elastic stresses. As shown in Figure 6, $s \, dA = dF$. From Figure 7 the moment of the four forces F_1, F_2, F_3, F_4 about neutral axis comprises the internal resisting moment at any cross section of a beam for which bending exists beyond the yield point of the material. Hence, $M_i = F_1 \bar{y}_1 + F_2 \bar{y}_2 + F_3 \bar{y}_3 + F_4 \bar{y}_4$, \bar{y} being the moment arm of the forces measured from the neutral axis.

Then

$$M_i = \int y_1 s_o \, dA + \int y_4 s_o \, dA + \int y s \, dA_2 + \int y s \, dA_3 \quad (1)$$

From this

$$M_i = A_1 s_o \bar{y}_1 + A_4 s_o \bar{y}_4 + \frac{s_o}{d_o} \int_0^{d_o} y^2 dA + \frac{s_o}{d_o'} \int_0^{-d_o'} y^2 dA$$

Therefore

$$M_i = s_o \bar{y}_1 A_1 + s_o \bar{y}_4 A_4 + \frac{s_o I_2}{d_o} + \frac{s_o I_3}{d_o'} \quad (1-a)$$

This is the fundamental equation for the internal resisting moment in a plastic-elastic cross section of a beam produced by bending. The cross section is assumed to have two axes of symmetry.

For the determination of the location of the neutral axis use can be made of the principle that at any cross section of the beam the sum of compressive forces equals the sum of the tensile forces.

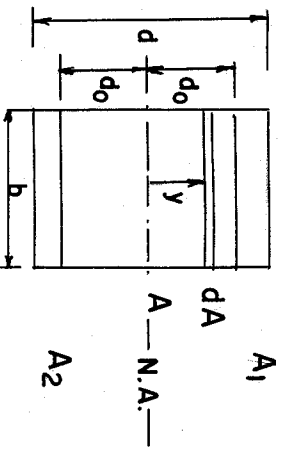


Fig. 9

Then

$$\int_0^{d_0} s dA - \int_0^{-d'_0} s dA = s_0 (A_2 - A_1) = \frac{s_0}{d_0} \int_{-d'_0}^{d_0} y dA \quad (2)$$

Since it is assumed that a plane section before bending remains plane after bending, the yield point in compression is equal to the yield point in tension, and the modulus of elasticity in compression equals modulus of elasticity in tension, the upper and lower limits of the integral $y dA$ are equal. Therefore the right side of equation 2 equals zero, and $A_1 - A_2 = 0$. Then $A_1 = A_2$ and since $d_0 = d'_0$, the neutral axis passes through the centroid of the whole section, for the width of the section is constant.

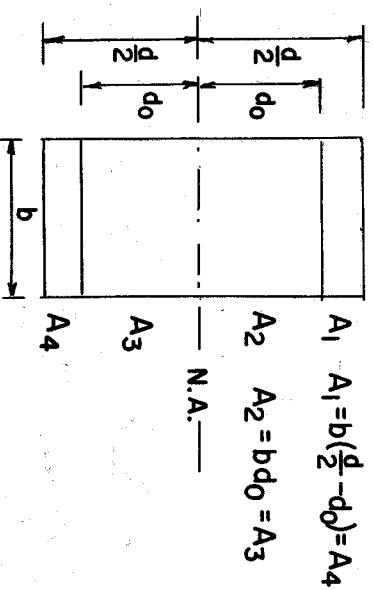


Fig. 10

Equation 1-a becomes $M_i = 2s_o \left[b\left(\frac{d}{2} - d_o\right) \right] \left[\frac{\frac{d}{2} + d_o}{2} \right] + \frac{2s_o}{3} b d_o^2$
 or $M_i = s_o b \left(\frac{d^2}{2} - d_o^2 \right) + \frac{2s_o b d_o^2}{3} = s_o b \left(\frac{d^2}{4} - \frac{d_o^2}{3} \right)$. Let $\frac{d_o}{\frac{d}{2}} = u$

$$M_i = s_o b \left(\frac{d^2}{4} - \frac{u^2 d^2}{12} \right) = s_o b d^2 \left(\frac{1}{4} - \frac{u^2}{12} \right).$$

This equation may be written

$$M_i = \frac{s_o b d^2}{6} \left(\frac{3}{2} - \frac{u^2}{2} \right) = M_o \left(\frac{3}{2} - \frac{u^2}{2} \right), \text{ where } M_o = \frac{s_o b d^2}{6}. \quad (4)$$

Equation 4 is the general expression for the internal resisting moment of a rectangular section when the beam is stressed above the yield point of the material. Note that the term M_o is the maximum moment of the section by theory of elasticity. Equation 4 may be obtained in a different manner by making use of the ideal stress-strain diagram, as shown in Figure 12.

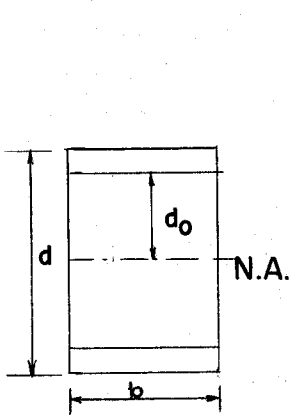


Fig. 11

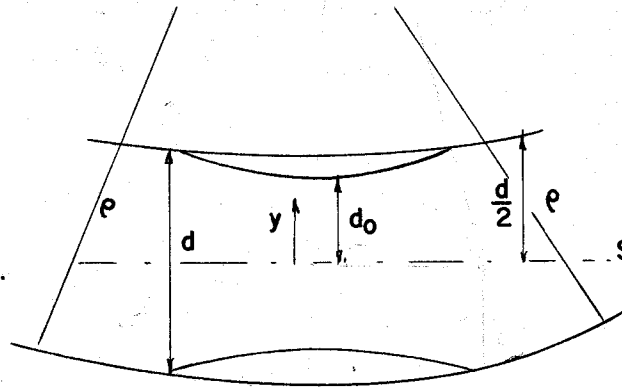


Fig. 12

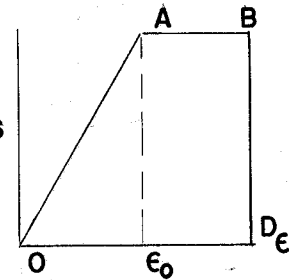


Fig. 13

At the distances y , d_o , and $\frac{d}{2}$ from the neutral axis represent the unit strains by ϵ , ϵ_o and ϵ_m respectively, Figure 12.

$$\frac{\epsilon}{y} = \frac{\epsilon_o}{d_o} = \frac{\epsilon_m}{\frac{d}{2}} = \frac{1}{\rho}, \quad (5)$$

where ρ is the radius of curvature of the beam. See any elementary strength of materials text. The internal moment

$$M_i = b \int_{-\frac{d}{2}}^{\frac{d}{2}} s y dy \quad (6)$$

Since $\frac{e}{y} = \frac{1}{\rho}$, $y dy = \rho^2 e de$, therefore

$$M_i = 2b \rho^2 \int_0^{\frac{\Delta}{2}} s e de ; \text{ where } 2 e_m = \Delta .$$

The term $\int_0^{\frac{\Delta}{2}} s e de$ represents the moment of the stress-strain area (CARD) about the stress axis. Evaluating the $\int_0^{\frac{\Delta}{2}} s e de$ and noting that

$$e_o = \frac{d_o}{\frac{d}{2}} = e_m u,$$

$$M_i = \frac{2bd^2}{\Delta^2} \left\{ s_o e^2 \left(\frac{1}{2} - \frac{u^2}{6} \right) = \frac{s_o b d^2}{6} \left(\frac{3}{2} - \frac{u^2}{2} \right) \right\}, \text{ and this is equation 4.}$$

Equation 6 may be written,

$$M_i = \frac{2bd^2}{\Delta^2} \int_0^{\frac{\Delta}{2}} s e de$$

$$M_i = \frac{1}{\rho} \cdot I \cdot \frac{1}{12 \Delta^3} \int_0^{\frac{\Delta}{2}} s e de .$$

Equation 6 then becomes

$$M_i = \frac{E_R I}{\rho} .$$

The term

$$\frac{1}{12 \Delta^3} \int_0^{\frac{\Delta}{2}} s e de \text{ is of the form of a modulus.}$$

In this form the equation is useful in the determination of deflections by the well known moment area method. The application is beyond the scope of this thesis.

D. Deflection Equations for a Beam of Rectangular Cross Section.
Concentrated Load at the Center of Span and Equal Moments at the Ends.

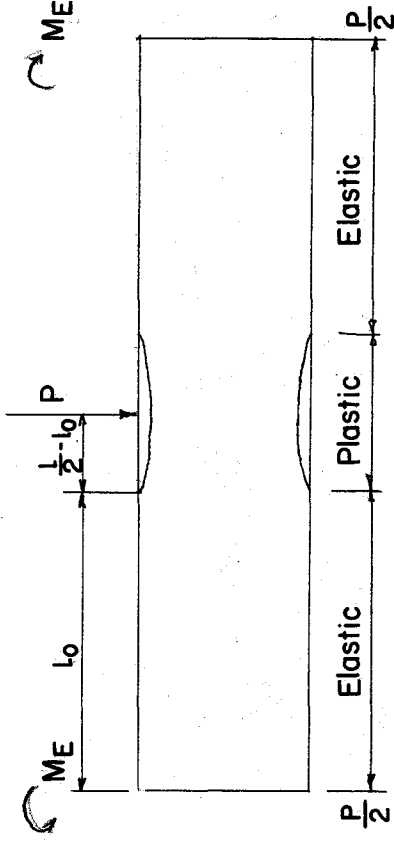


Fig. 14

If a beam of rectangular cross section is loaded as shown in Figure 14 and is stressed above the yield point of the material, by bending, two distinct parts of the beam to the left of the load are observed. The part between the left support and point l_0 distance from the support is acting elastically. A part between l_0 and the center is acting plastically. Two differential equations are necessary in order to express the deflection curve of the beam. The origin of coordinates is at the left support. For the elastic part of the beam the well known differential equation for the elastic line from the theory of elasticity is

$$EI \frac{d^2 y}{dx^2} = -M \quad (7)$$

and for the plastic part the differential equation is given by

$$-\frac{d^2 y}{dx^2} = \frac{1}{\rho}, \text{ from equation 5.}$$

For the elastic part

$$EI \frac{d^2 y}{dx^2} = \frac{Px}{2} - M_E, \quad (8)$$

where M_E is the end moment. A single integration of this equation yields

$$EI \frac{dy}{dx} = \frac{Px^2}{4} - M_E x + C_1 \quad (9)$$

Then

$$EI y = \frac{Px^3}{12EI} - \frac{M_E x^2}{2EI} + C_1 x + C_2 \quad (10)$$

At the left support $x = 0$, $y = 0$, therefore $C_2 = 0$.

Equation 9 may be written

$$\frac{dy}{dx} = \frac{Px^2}{4EI} - \frac{M_E x}{EI} + \frac{C_1}{EI} \quad (9-a)$$

Equation 10 may be written

$$y = \frac{Px^3}{12EI} - \frac{M_E x^2}{2EI} + \frac{C_1 x}{EI} \quad (10-a)$$

Equations 9-a and 10-a are similar to those for purely elastic action of the beam, with $\frac{C_1}{EI}$ modified in order to take into account the effect of the plastic action of the center part of the beam.

For the length of the beam between $\frac{l}{2}$ and l_0

$$\text{Write equation 5, } \frac{d^2 y}{dx^2} = \frac{1}{\rho} = \frac{\epsilon_0}{d_0} \quad (11)$$

$$\frac{d^2 y}{dx^2} = \frac{s_0}{Ed_0}, \text{ since } s_0 = E\epsilon_0.$$

It should be noted that d_0 is a function of x (decreasing towards the left support), and if $\frac{d_0}{2}$ is expressed as "u" equation 11 may be written,

$$\frac{d^2 y}{dx^2} = \frac{s_0}{E \frac{d_0}{2}} = \frac{2s_0}{Ed} u^{-1} = Au^{-1} \quad (11a)$$

where A represents $\frac{2s_0}{E}$.

From condition of statics the external bending moment is equal to the internal resisting moment, M_1 . The external moment is given by the right side of equation 8 and the internal moment is given by equation 4.

Hence,

$$\frac{Px}{2} - M_E = M_0 \left(\frac{3}{2} - \frac{u^2}{2} \right),$$

and therefore

$$u^2 = 3 + \frac{2ME}{M_0} - \frac{P}{M_0} x. \quad (12)$$

If $3 + \frac{2ME}{M_0} = K_1$, and $\frac{P}{M_0} = K_2$, equation 12a may be written

$$u = (K_1 - K_2 x)^{\frac{1}{2}}. \quad (12-a)$$

This is the equation of the plastic boundary of the center portion of the beam.

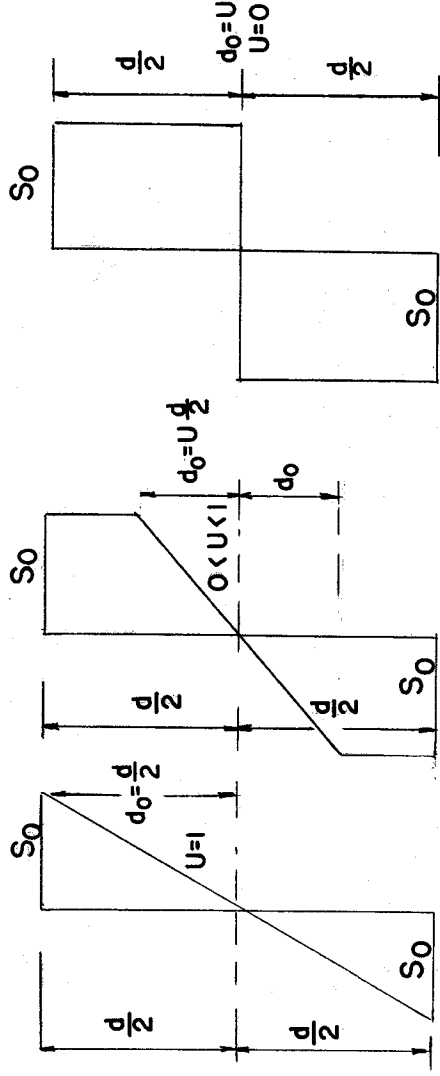


Fig. 15

Fig. 16

Fig. 17

The value of "u" is never greater than 1, or less than zero. When it is equal to 1 the beam has just begun to yield. See Figure 15. When it is greater than zero and less than unity the stress distribution is as shown by Figure 16, and when the whole of the beam is stressed to the yield point "u" equals zero, and the stress distribution diagram is represented by two rectangles as shown in Figure 17.

Substituting equation 12-a in equation 11-a

$$\frac{d^2 y}{dx^2} = A(K_1 - K_2 x)^{-\frac{1}{2}} \quad (13)$$

Integration of equation 13 yields,

$$\frac{dy}{dx} = \frac{-2A}{K_2}(K_1 - K_2 x)^{\frac{1}{2}} + C_3 \quad (14)$$

At the center $\frac{dy}{dx} = 0$. Therefore, $C_3 = \frac{2A}{K_2}(K_1 - K_2 \frac{1}{2})^{\frac{1}{2}}$, therefore equation 14 may be written

$$\frac{dy}{dx} = \frac{-2A}{K_2}(K_1 - K_2 x)^{\frac{1}{2}} + \frac{2A}{K_2}(K_1 - K_2 \frac{1}{2})^{\frac{1}{2}} \quad (14-a)$$

Integration of equation 14-a yields,

$$y = \frac{4}{3} \frac{A}{K_2^2}(K_1 - K_2 x)^{\frac{3}{2}} + \frac{2A}{K_2}(K_1 - K_2 \frac{1}{2})^{\frac{1}{2}} x + C_4 \quad (15)$$

In order to determine the shear in this section of the beam, write equation 13

$$\frac{d^2 y}{dx^2} = \frac{2s_o \frac{bd^2}{12}}{Ed \frac{bd^2}{12}} u^{-1} = \frac{M_o u^{-1}}{EI} \quad (16)$$

Therefore

$$EI \frac{d^2 y}{dx^2} = M_o u^{-1} \quad (16-a)$$

This is an equation analogous to equation 7, $M_0 u^{-1}$ corresponding to $\frac{Px}{2} - M_E$. From strength of materials the shear, $V = \frac{dM}{dx}$. Therefore

$$\begin{aligned} V &= \frac{d}{dx} (M_0 u^{-1}) = \frac{M_0 u}{2} (K_1 - K_2 x)^{-\frac{1}{2}} (-K_2) \\ &= \frac{M_0 K_2 u}{2(K_1 - K_2 x)^{\frac{1}{2}}} = \frac{M_0 K_2}{2} = \frac{M_0}{2} \cdot \frac{P}{M_0} = \frac{P}{2}. \end{aligned}$$

This is the same value of the shear when determined by obtaining the first derivative of equation 7.

E. Determination of Constants of Integration

At the point when the material has just begun to yield the abscissa $x = l_0$, $u = 1$. At this point the slope of the two parts of the beam are common, and the deflections are equal. From equations 14-a and 9-a

$$\begin{aligned} \frac{Px^2}{4EI} - \frac{M_E x}{EI} + \frac{C_1}{EI} &= \frac{-2A}{K_2} (K_1 - K_2 x)^{\frac{1}{2}} + \frac{2A}{K_2} (K_1 - K_2 l_0)^{\frac{1}{2}} \\ \frac{C_1}{EI} &= \frac{Pl_0^2}{4EI} + \frac{M_E l_0}{EI} - \frac{2A}{K_2} (K_1 - K_2 l_0)^{\frac{1}{2}} + \frac{2A}{K_2} (K_1 - K_2 l_0)^{\frac{1}{2}} \end{aligned}$$

Thus equation 9, of slope, may be written

$$\frac{dy}{dx} = \frac{Px}{4EI} - \frac{M_E}{EI} + \frac{M_E l_0}{EI} - \frac{Pl_0^2}{4EI} - \frac{2A}{K_2} (K_1 - K_2 l_0)^{\frac{1}{2}} + \frac{2A}{K_2} (K_1 - K_2 l_0)^{\frac{1}{2}} \quad (17)$$

The equation of the deflection curve in the range $0 < x < l_0$ is

$$y = \frac{Px^3}{12EI} - \frac{M_E x^2}{2EI} + \frac{M_E l_0 x}{EI} - \frac{Pl_0^2 x}{4EI} + \frac{2A}{K_2} (K_1 - K_2 l_0)^{\frac{1}{2}} x + \frac{2A}{K_2} (K_1 - K_2 l_0)^{\frac{1}{2}} x \quad (18)$$

The equation 14, of slope, becomes

$$\frac{dy}{dx} = -\frac{2A}{K_2} (K_1 - K_2 x)^{\frac{1}{2}} + \frac{2A}{K_2} (K_1 - K_2 l_0)^{\frac{1}{2}} \quad (19)$$

Equating the deflections of equations 10-a and 15, substituting the value for C_1 , and solving for C_4

$$C_4 = \frac{Pl_o^3}{12EI} - \frac{M_E l_o^2}{2EI} + \left\{ \frac{M_E l_o}{EI} - \frac{Pl_o^2}{4EI} - \frac{2A}{K_2}(K_1 - K_2 l_o)^{\frac{1}{2}} + \frac{2A}{K_2}(K_1 - K_2 l_o)^{\frac{1}{2}} - \frac{4}{3} \frac{A}{K_2^2}(K_1 - K_2 l_o)^{\frac{3}{2}} + \frac{2A}{K_2}(K_1 - K_2 l_o)^{\frac{1}{2}} \right\} l_o.$$

Equation 15 in the range $l_o \leq x \leq \frac{1}{2}$ may be written

$$y = \frac{4}{3} \frac{A}{K_2^2}(K_1 - K_2 x)^{\frac{3}{2}} + \frac{2A}{K_2}(K_1 - K_2 l_o)^{\frac{1}{2}} x - \frac{2A}{K_2^2}(K_1 - K_2 l_o)^{\frac{1}{2}} l_o + \frac{Pl_o^3}{12EI} - \frac{M_E l_o^2}{2EI} + \left\{ \frac{M_E l_o}{EI} - \frac{Pl_o^2}{4EI} - \frac{2A}{K_2}(K_1 - K_2 l_o)^{\frac{1}{2}} + \frac{2A}{K_2}(K_1 - K_2 l_o)^{\frac{1}{2}} \right\} l_o - \frac{4}{3} \frac{A}{K_2^2}(K_1 - K_2 l_o)^{\frac{3}{2}} \quad (20)$$

F. Deflection Equation for Post

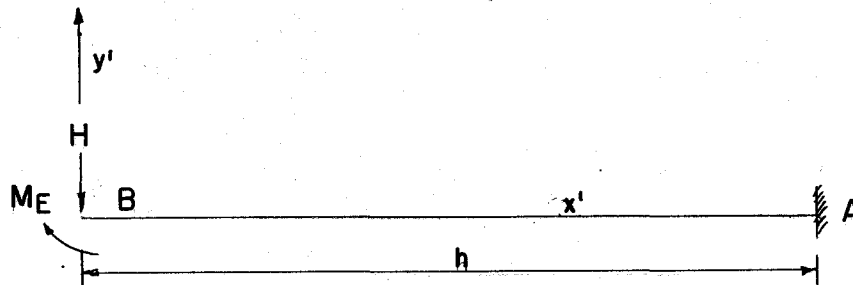


Fig. 19a

The elastic curve for the post of a fixed-end rigid frame when the material is not stressed above the yield point may be obtained by two integrations of the well known differential equation $\frac{d^2 y}{dx^2} = -\frac{M}{EI}$.

$$EIy'' = M_E - Hx' \quad (21)$$

$$EIy' = M_E x' - \frac{Hx'^2}{2} + C_1 \quad (21a)$$

From the boundary condition that $x' = h$, $\frac{dy'}{dx} = 0$.

$$C_1 = -M_E h + \frac{Hh^2}{2}$$

Therefore,

$$\frac{dy}{dx} = \frac{M_E x}{EI} - \frac{Hx^2}{2EI} - \frac{M_E h}{EI} + \frac{Hh^2}{2EI} \quad (22)$$

$$y = \frac{M_E x^2}{2EI} - \frac{Hx^3}{6EI} - \frac{M_E h x}{EI} + \frac{Hh^2 x}{2EI} + C_2$$

From the boundary condition $x = 0, y = 0, C_2 = 0$

$$y = \frac{M_E x^2}{2EI} - \frac{Hx^3}{6EI} - \frac{M_E h x}{EI} + \frac{Hh^2 x}{2EI} \quad (23)$$

Equation 23 is the equation of the elastic line of the post. At $x = h,$

$y = 0$, therefore $M_E = \frac{2}{3} Hh.$

$$\text{At } x = 0, \frac{dy}{dx} = - \frac{M_E h}{4EI} \quad (24)$$

Now the quantity M_E is one that is statically indeterminate in a rigid frame and in order to determine this quantity for any load P , a known height and width of frame, and moment of inertia of the cross section, use will be made of the fact that at the point B of the frame the angle changes of the post and beam are the same. At point B one value of $\theta_B = \frac{dy}{dx}(\theta_B$ denoting angle change at B) is found by determining the value of equation 17 at $x = 0$. This is the value for the beam. For the frame θ_B is given by equation 24. Observe that the frame post is acting elastically and its differential equation of the elastic curve is equation 21.

Hence,

$$- \frac{M_E h}{4EI_1} = \frac{M_E l_0}{EI_2} - \frac{Pl_0^2}{4EI_2} - \frac{2A}{K_2}(K_1 - K_2 l_0)^{\frac{1}{2}} + \frac{2A}{K_2}(K_1 - K_2 l_0)^{\frac{1}{2}}$$

Solving for M_E

$$M_E = \frac{-\frac{Pl_o^2}{4I_2} + E \frac{2A}{K_2} \left\{ (K_1 - K_2 \frac{1}{2})^{\frac{1}{2}} - (K_1 - K_2 l_o)^{\frac{1}{2}} \right\}}{\frac{h}{4I_1} + \frac{l_o}{I_2}} \quad (25)$$

Here I_1 and I_2 represent the moment of inertia of the post and beam, respectively. The quantities h and l represent the height of post and length of beam, respectively. Equation 25 is very similar to the well known equation for M_E when the frame is acting elastically,

$$M_E = \frac{\frac{P(\frac{1}{2})^2}{4I_2}}{\frac{h}{4I_1} + \frac{\frac{1}{2}}{I_2}} .$$

Two methods are available for the determination of the value of M_E in the frame for any applied load. The equation 25 contains two quantities that change with P , the load at the center. M_E and l_o are related by the equation $3 + \frac{2M_E}{M_o} = K_1$. Obviously as equation 25 is written M_E is not explicitly determined, since K_1 appears on the right side of the equation. Moreover the quantity l_o is a function of the load for the edge of the plastic boundary spreads toward the supports. In order to satisfy the statics of the beam member, the equation $M_E = \frac{Pl_o}{2} - M_o$ is obtained. This equation shows the relation between P and l_o . Substitution of the value l_o , from this equation, into equation 25 and simplifying the result is a fourth degree equation. The resulting fourth degree equation may be solved by any of the standard methods found in any text of Theory of Equations. The development of the equation follows. M is M_E in the following equations.

Write equation 25

$$-\frac{Mh}{4EI} - \frac{Ml_o}{4EI_2} - \frac{Pl_o^2}{4EI_2} - \frac{2A}{K_2}(K_1 - K_2l_o)^{\frac{1}{2}} + \frac{2A}{K_2}(K_1 - K_2\frac{l}{2})^{\frac{1}{2}}$$

Then,

$$M(\frac{h}{4I_1} + \frac{l_o}{I_2}) - \frac{Pl_o^2}{4EI_2} - \frac{2A}{K_2} = -\frac{2A}{K_2}(K_1 - K_2\frac{l}{2})^{\frac{1}{2}}$$

Square both sides of this equation

$$\left\{ M(\frac{h}{4I_1} + \frac{l_o}{I_2}) - \frac{Pl_o^2}{4EI_2} - \frac{2A}{K_2} \right\}^2 = \left(\frac{2A}{K_2} \right)^2 (K_1 - K_2\frac{l}{2})$$

Substitute the value of l_o , and expand the left side, and simplify the result.

$$\begin{aligned} & \frac{9M^4}{P^2I_2^2} - \frac{3M^3h}{2PI_1I_2} - \frac{24M^3M_o}{P^2I_2^2} + \frac{2M^2M_o h}{PI_1I_2} + \frac{2M^2M_o h}{PI_1I_2} + \frac{16M^2M_o^2}{P^2I_2^2} + \frac{M^2h^2}{16I_1^2} + \frac{6M^2M_o}{P^2I_2^2} \\ & + \frac{12AEM^2}{PI_2K_2} - \frac{M_o h M}{2PI_1I_2} - \frac{8M_o^2 M}{P^2I_2^2} - \frac{AEhM}{I_2I_1} - \frac{16AEM_o M}{PI_2K_2} - \frac{8A^2E^2M}{K_2^2M_o} \\ & = -\frac{4AEM_o}{PI_2K_2} + \frac{4A^2E^2}{K_2^2} \left(3 - \frac{Pl}{2M_o} \right) - \frac{M_o^2}{P^2I_2^2} - \frac{4A^2E^2}{K_2^2} \end{aligned} \quad (26)$$

Dividing through by the coefficient of M^4 after collecting all terms of same degree, the equation may now be written

$$M^4 - bM^3 + cM^2 - dM - e = 0, \quad (27)$$

where

$$\begin{aligned} b = & \left(\frac{3}{18} \frac{hP}{PI_1I_2} + 24 M_o \right); \quad c = \frac{2}{9} \frac{M_o h}{PI_1I_2} + \frac{16}{9} M_o^2 + \frac{1}{144} P^2 h^2 \left(\frac{I_2}{I_1} \right)^2 \\ & + \frac{2}{3} M_o P^2 I_2^2 + \frac{4}{3} \frac{PAEI_2}{K_2}, \end{aligned}$$

$$d = \frac{M_o h P I_2}{18 I_1} + \frac{8}{9} M_o^2 P + \frac{11}{9} AEhP^2 + \frac{16}{9} AEM_o I_2 P + \frac{8}{9} \frac{A^2 E^2 I_2^2 P^2}{9 K_2^2 M_o},$$

$$\text{and } e = \frac{P^2 I_2^2}{9} \left\{ \frac{4A^2 E^2}{K_2^2} \left(2 - \frac{Pl}{2M_o} \right) - \frac{4AE}{PI_2K_2} - \frac{M_o^2}{P^2 I_2^2} \right\}.$$

Following is the approximation method for the determination of M_E ,
when $P = 4430$ lb.

$$M_E = \frac{\frac{Pl_o^2}{4I_2} + \frac{2AE}{K_2} \left[(K_2 - K_{22} \frac{1}{2})^{\frac{1}{2}} - (K_1 - K_2 l_o)^{\frac{1}{2}} \right]}{\left(\frac{h}{4I_1} + \frac{l_o}{I_2} \right)} \quad (25)$$

Assume $l_o = 20$ in.

$$K_2 = \frac{P}{M_o} = 0.159$$

$$K_1 = 1 + 3.18 = 4.18$$

$$(K_1 - K_{22} \frac{1}{2})^{\frac{1}{2}} = 0.574$$

$$M_E = \frac{\frac{4430 \times 50^2}{4 \times 0.978} + 156,000}{18.4 + 20.5} = 15,250; \frac{2A}{K_2} = 0.0122$$

$$\text{From statics } M_E = \frac{4430}{2} \times 20 = 27,800 = 16,500 \text{ " \#}$$

Second Assumption

Assume $l_o = 19.7$ in.

$$M_E = \frac{\frac{Pl_o^2}{4I_2} + 2AE \left\{ (K_1 - K_{22} \frac{1}{2})^{\frac{1}{2}} - (K_1 - K_2 l_o)^{\frac{1}{2}} \right\}}{\frac{h}{4I_1} + \frac{l_o}{I_2}}$$

$$= \frac{440,000 + 70,000}{18.4 + 20.1} = 15,850$$

$$\text{From statics } M_E = \frac{4430}{2} \times 19.7 = 27,800 = 15,900 \text{ " \#}$$

G. Examples Making Use of Deflection Equations

Following is an example of solution for y at $x = 24"$ and at $x = 15"$,
the angle change, and frame spread.

Given the load $P = 4430$ lb., determine the deflection at the center of the span of the beam member B B'. The deflection may be determined by substituting directly into equation 20. A procedure for the determination of the quantities would follow in the manner: (1) Determine M_E by the approximation method as shown by the example on page 34. Next determine

$$K_2, K_1, \frac{2A}{K_2}, \frac{4A}{3K_2^2}, \text{ and } l_0.$$

$$K_2 = \frac{P}{M_0} = 0.1590^{1-1}; K_1 = 3 + \frac{2M_E}{M_0} = 4.13; \frac{2A}{K_2} = 0.0122;$$

$$\frac{4A}{3K_2^2} = 0.0515 \text{ in.}; l_0 = 19.70 \text{ in.}$$

From equation 20, $y = -0.0438 + 0.0310 + 2140 - 2840 - 1135 = -0.1963$ in.

The experimental value for this load is -0.2095 in.

The deflection of the member at $x = 15$ in. may be determined by substituting into equation 18. All the quantities in this equation are known, following the solution for y at $x = 24$ in. For this load $y = +0.0432 - 0.0618 - 0.1468 = -0.1654$ in. The experimental value is -0.1593 in.

To determine the angle change substitute directly into equation 24.

$$\phi = \frac{dy}{dx} = -\frac{M_E h}{4EI_1} = -0.00970 \text{ radians.}$$

The experimental value is 0.00956 . The frame spread may be determined by direct substitution into equation 23. This equation reduces to $y = \frac{192 M_E}{EI}$, when $x' = 24$ in. Note that H is $\frac{3}{2} \frac{M_E}{h}$. Therefore $y = -0.1055$ in., and the observed value is equal to $\frac{1189 + 1016}{2} = -1102$ in.

The beam deflections may be determined in the same manner as for the frame.

III. EXPERIMENTAL

A. Scope

The experimental phase of this investigation included the measurement of the deflection, angle changes, strains, and the measurement of deflections of a beam. The frame was a 'U' type of rigid frame made of mild steel bars $2\frac{1}{2}$ in. by $3\frac{1}{4}$ in. in cross section. All joints were welded. The bottom ends of the posts of the frame were welded to two 15 in. 33.9 lb. channel sections; therefore the fixed end conditions were assumed to be realized at the lower joints. The members of the frame were assumed to be straight throughout the length and of uniform cross section. The posts heights were 72 in., measured from the center of the beam member to the top of the welded joints at the bottom of the posts. The beam member length was 48 in., measured from center to center of the post. Plates 2 and 3 show the frame under test.

The beam tested was 72 in. in length and supports were placed so that the beam would have an overhang of 12 in. at each support. As has been mentioned in the introductory chapter of this thesis the beam was so tested as to simulate the action of the beam member of the frame when all the members of the frame were acting elastically. The beam was of the same cross section and material as the beam member of the frame. The overhanging part of the beam was used to apply equal end moments to the beam simultaneously with a center load. Plate 1 shows the beam under test.

Figure 18 shows the beam under test and the dimensions for the loading beams. The algebraic equations developed in this investigation required the determination of the modulus of elasticity and yield point of the material in the frame and beam. The experimental determination was made in the laboratory of the Department of Theoretical and Applied Mechanics. A test specimen cut from the material of which the beam member of the frame was made was tested in tension. The specimen tested was 16 in. long and strains were measured over an 8 in. gage length. The stress-strain diagram from the data observed is shown in the appendix.

From the same stock as the tensile test specimen was cut a beam 62 in. long. This beam was tested in flexure over a span length of 48 in. A federal dial was used to measure the center deflection of the beam. The beam was loaded in increments of load producing a center deflection of one-thousandth of one inch. The maximum load applied was approximately 75 per cent of the yield point load. The yield point load is here defined as the load which produces a stress in the outside fiber equal to the yield point stress. The results of these data are shown in Figures 34 and 35 of the appendix. The experimental values of 'E' by tensile test were found to be 29.5×10^6 . By the flexure test 'E' was found to be 29.6×10^6 pounds per square inch. The yield point stress was found to be 35,600 pounds per square inch.

B. The Beam Test

The beam was tested in a Tinius Olsen Machine of 20,000 lb. capacity

in the laboratory of Theoretical and Applied Mechanics. The dials were placed at seven points along the beam as shown by Plate 1. These measurements were made by two men, one reading the applied loads and one reading the Federal dials. The beam was set in the machine and the loading beams placed as shown in Plate 1. Next, the beam was loaded in increments of load that corresponded to a center deflection of ten one-thousandths of an inch. The load was increased to approximately 70 per cent of the load that causes the center of the beam to be stressed above the yield point. The dials were read for all these increments of loading and unloading. The main purpose of these preliminary tests was to check the symmetry of the loading upon the beam and the applied moments at the ends.

After satisfying the condition of symmetry, the dials at points 30, 36, and 42 in. from the left support were changed to points 27, 33, and 39 in. from the left support. Up to approximately the yield point of the material the beam was loaded in the same manner as used in the symmetry tests. At this point the load was read simultaneously with the center dial only. The beam was allowed to stand for approximately 3 minutes and the load and all dials were read. There was naturally a decrease in the load reading and an increase in the center dial reading.

The load at the center was increased to approximately 140 per cent of the yield point load at the center, and the deflections and loads measured in the same manner as described above. After the yield point load was reached the beam was unloaded in increments of approximately 20 to 25 ten-thousandths of an inch center deflection and all dials were read, until the beam was unloaded.

Next, the beam was allowed to remain unloaded for a period of 24 hours and it was again loaded and the same procedure was repeated as described in the first loading tests. The maximum load was approximately 152 per cent of the original yield point load. Following this the beam was allowed to stand unloaded for a period of approximately 24 hours and it was again loaded and unloaded in the same manner as for the two previous loadings. The maximum load was the same as for the second loading.

C. The Frame Test

The frame was tested in the same machine as the beam. Figure 19 shows the dimensions of the frame. These tests were made in order to measure the deflections along the beam member of the frame, the angle changes at the joints of the posts and beam, the strains at four points along the beam member, and the spread of the frame at a point 24 in. below the center line of the beam. Federal dials of the same range and accuracy as those used in the beam tests were used to measure the deflections. A clinometer of gage length of $3 \frac{7}{8}$ in. was used to measure indirectly the angle changes. Huggenberger tensometers were used to measure the strains and Federal dials were used to measure the frame spread.

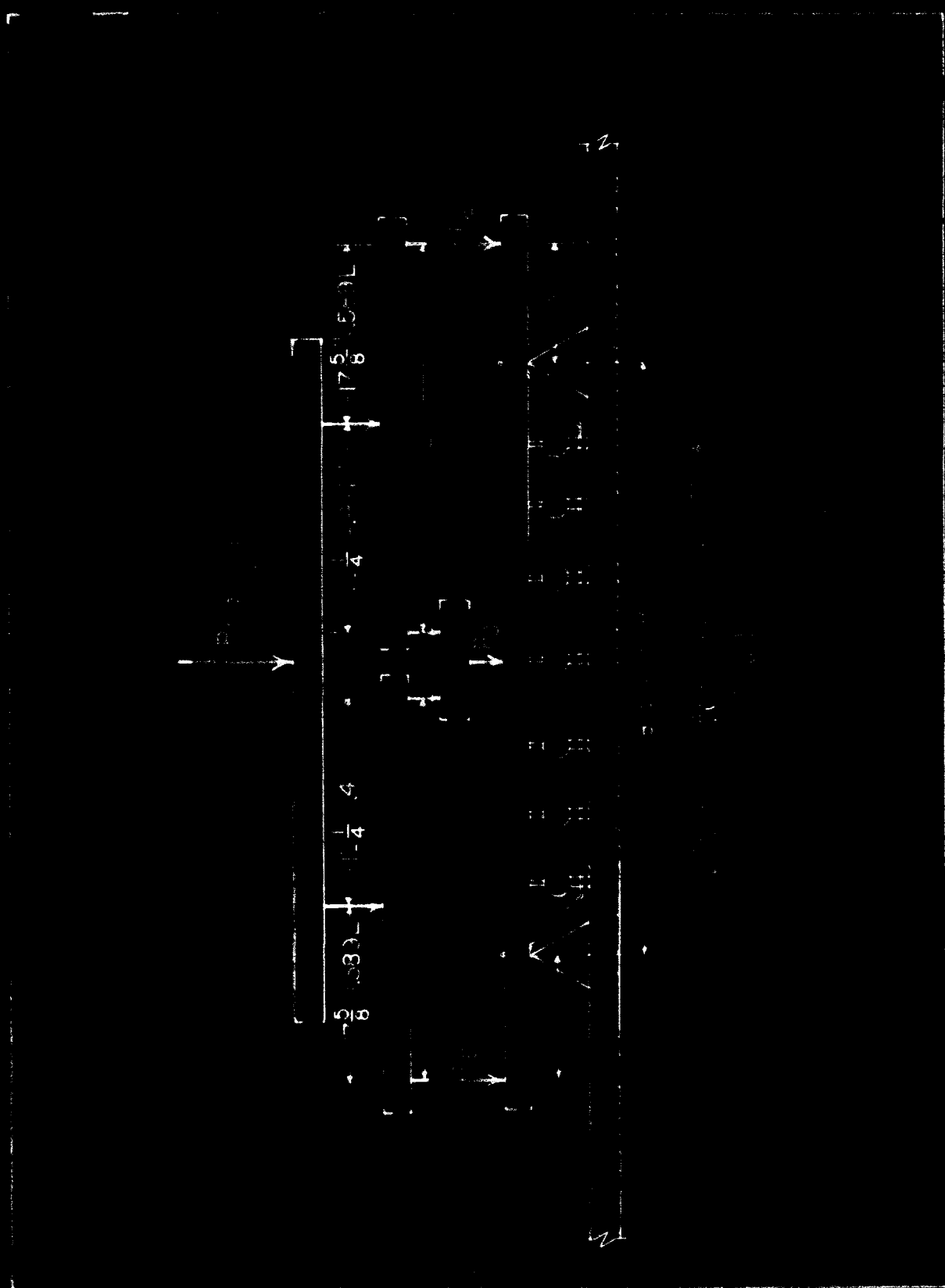
A symmetry test was made for the frame in the same manner as the one for the beam. In addition to this the clinometers were read for each load. The terminal load was approximately 75 per cent of the yield point load.

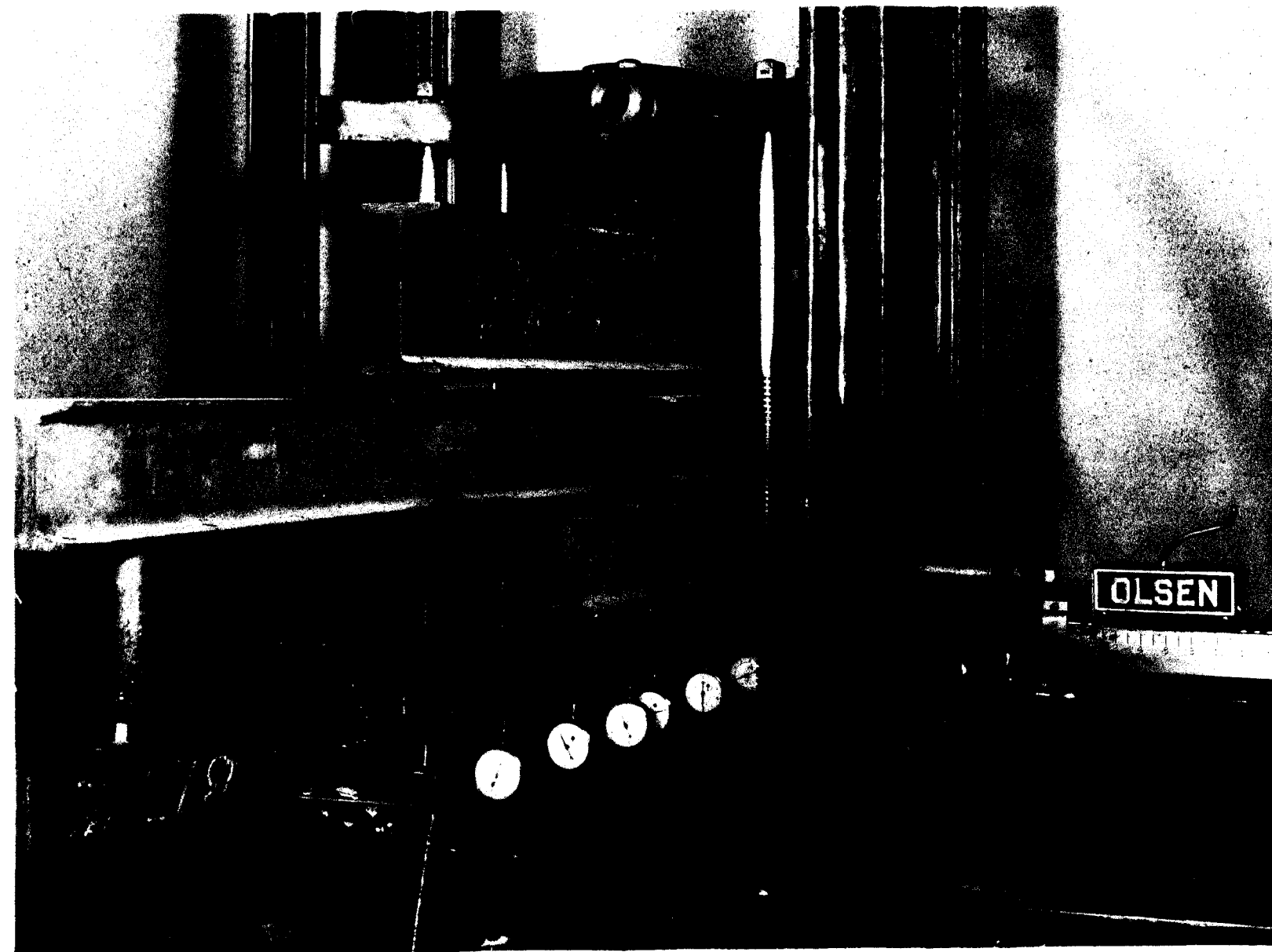
After the symmetry was satisfied the frame was loaded and the dials of all the test instruments were read for a load increment of approximately

one ten-thousandth of an inch. This procedure was followed until the load at the center reached approximately 90 per cent of the yield point load. At this point the dials for measuring the deflections and the frame spread were set for new zero readings.

After this the frame was loaded with increments of load corresponding to a center deflection of approximately ten one-thousandths of an inch. After the yield point of the material was reached at the center, the strain at the center and the center deflection dial were read simultaneously with the load. After approximately 4 minutes, the load, strains, clinometer dial, frame spread and deflection dials were read. Following this the load was increased to cause ten one-thousandths of an inch deflection at the center and the load and strain at the center were read simultaneously. All the other dials were read in the same order as for the previous load until the maximum load was reached. After reaching the maximum load the frame was unloaded in the same manner as was the beam, and all the instruments were read for all unloading values of the center load.

The frame was allowed to stand unloaded for approximately six days after which it was tested in the same manner as during the first loading. The maximum load for this test was approximately 45 per cent greater than the yield point load. The frame was unloaded in the same manner as in the first loading tests.





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Plate I

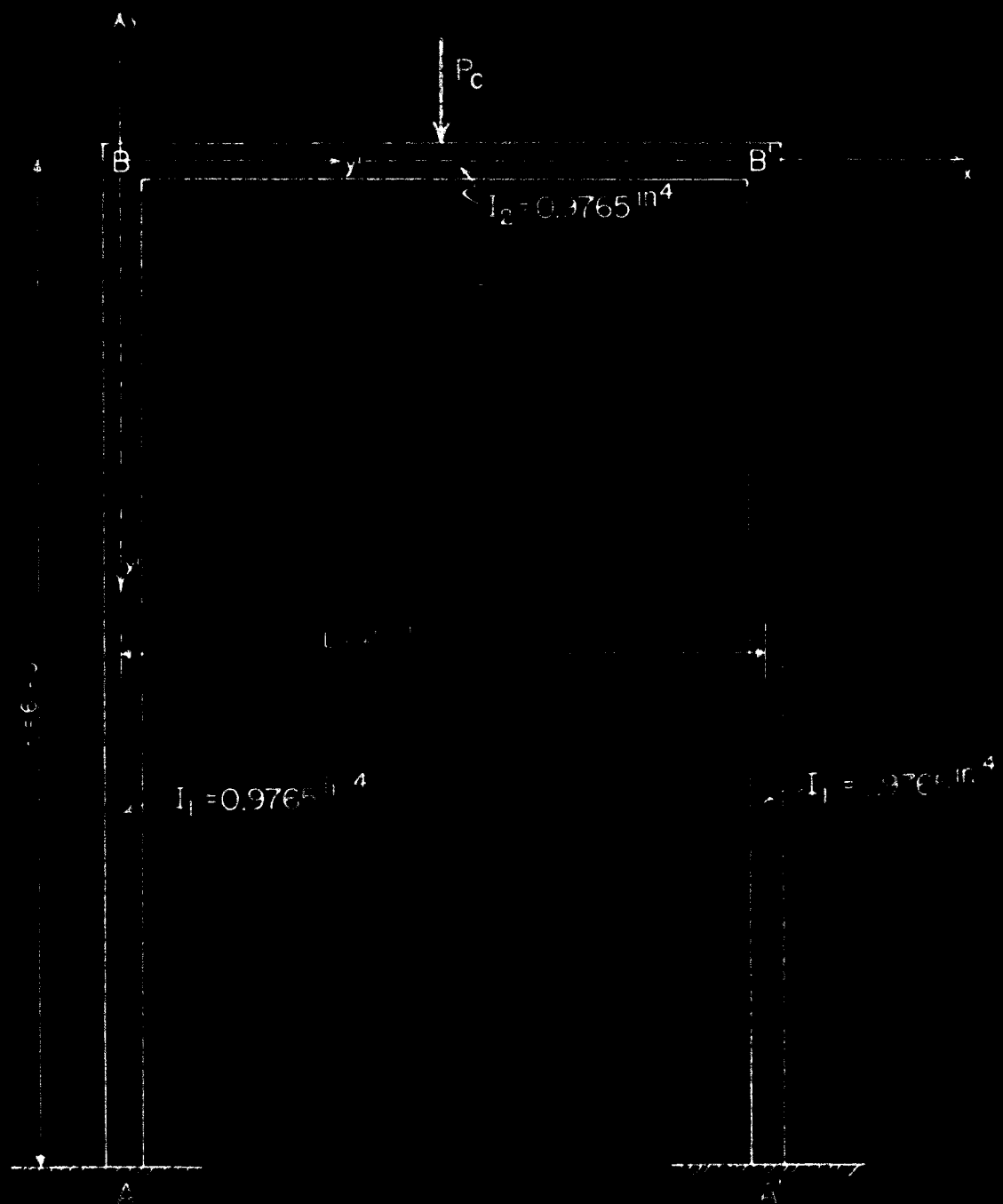


Diagram of "U" Frame
Fig. 19

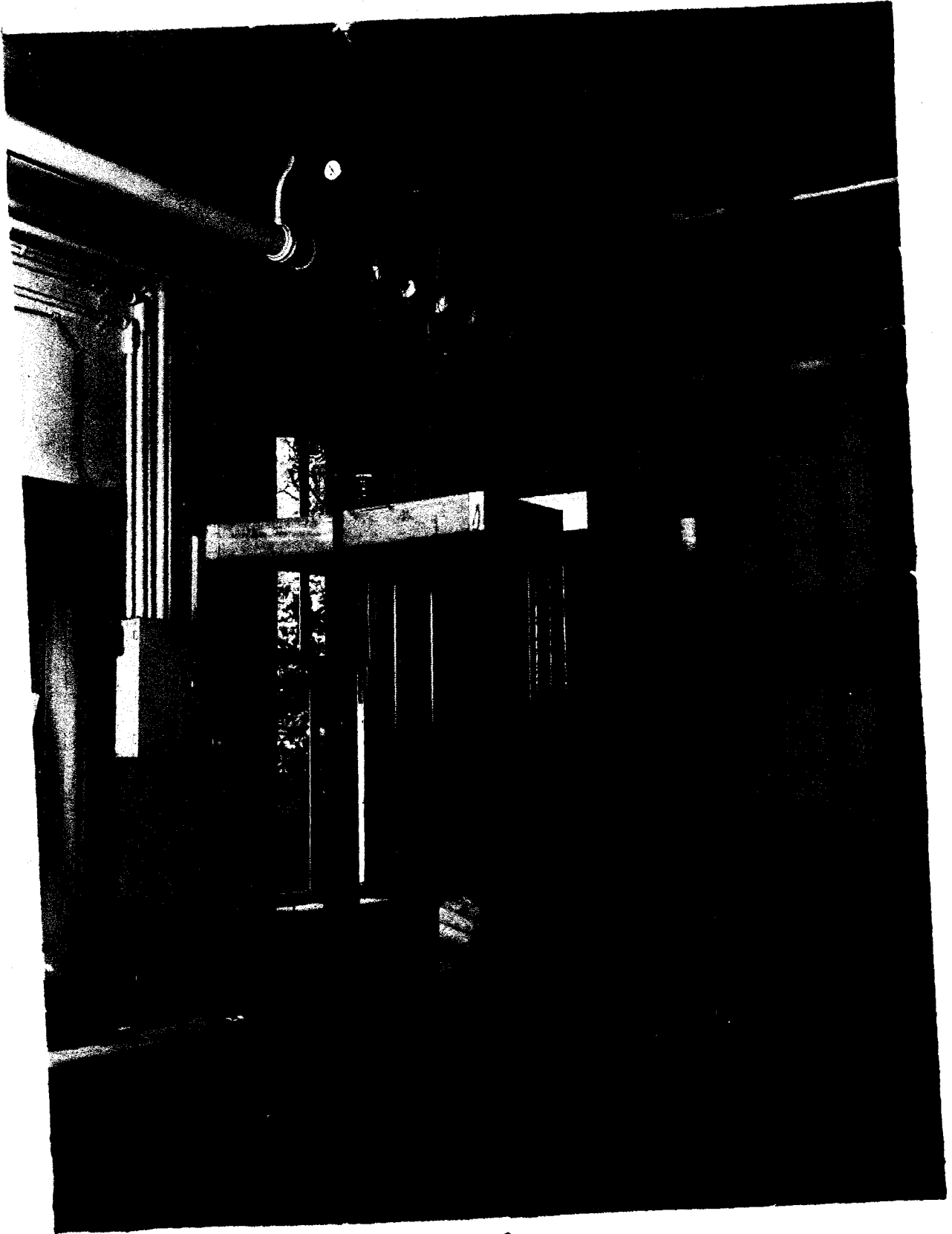
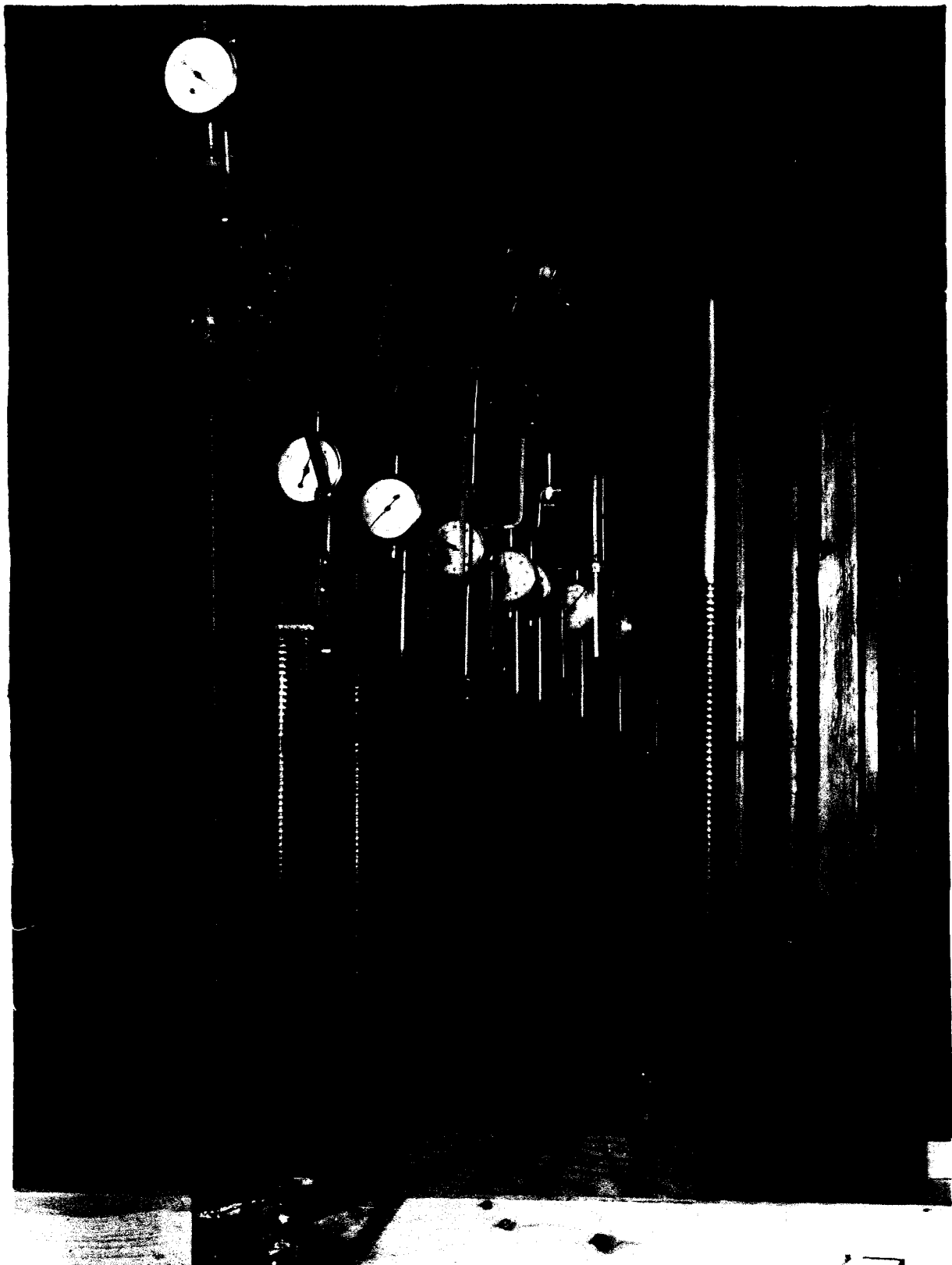
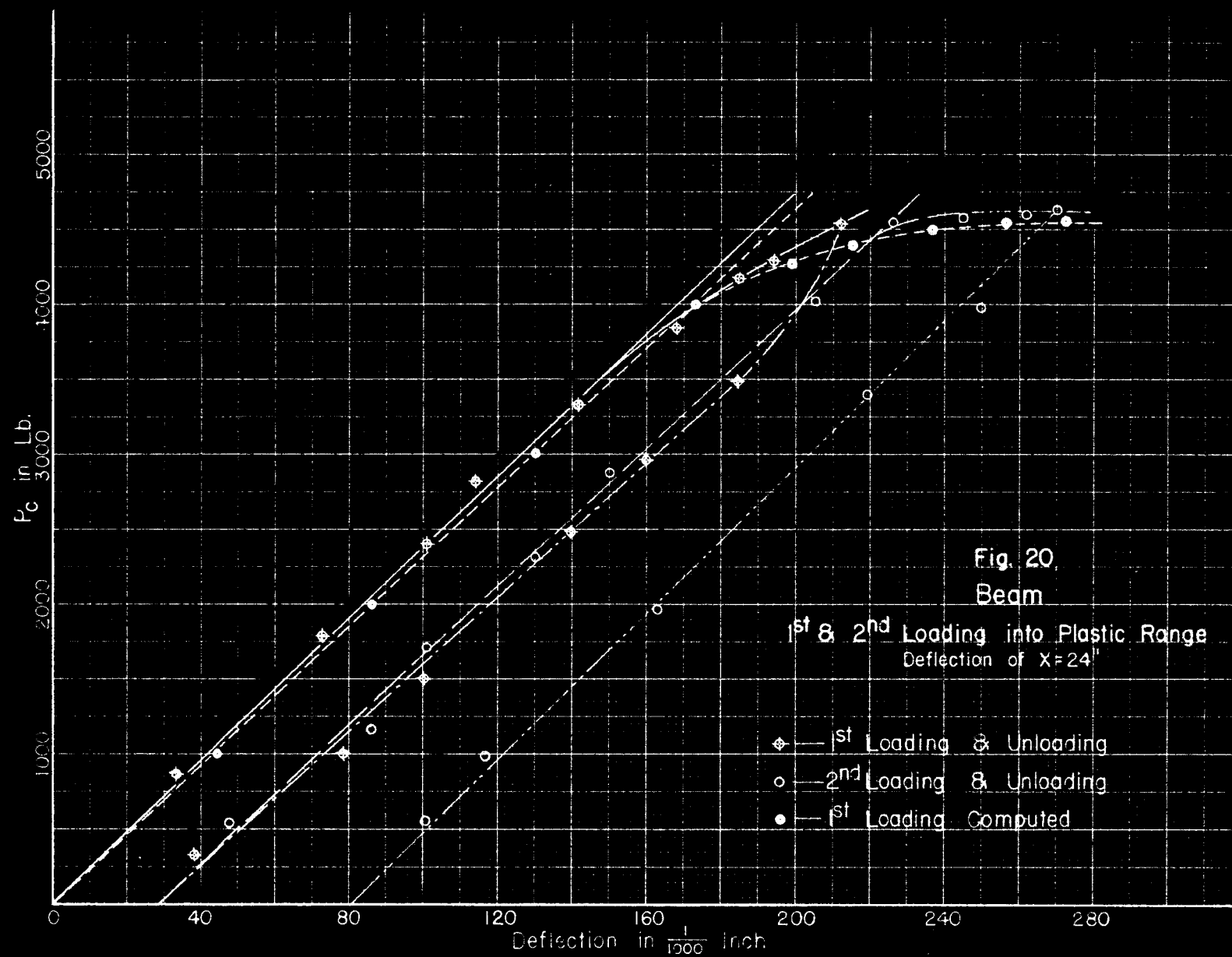
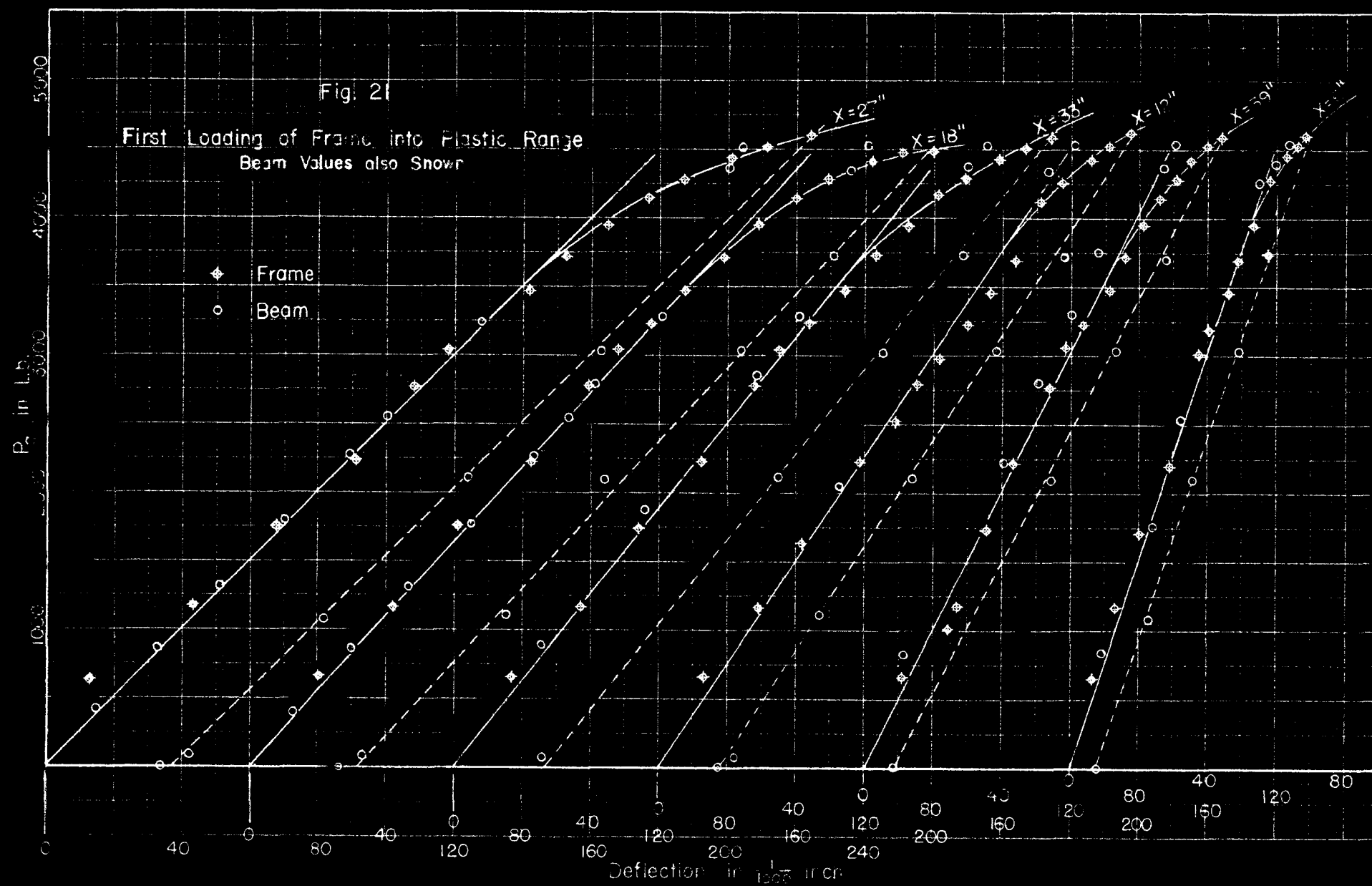


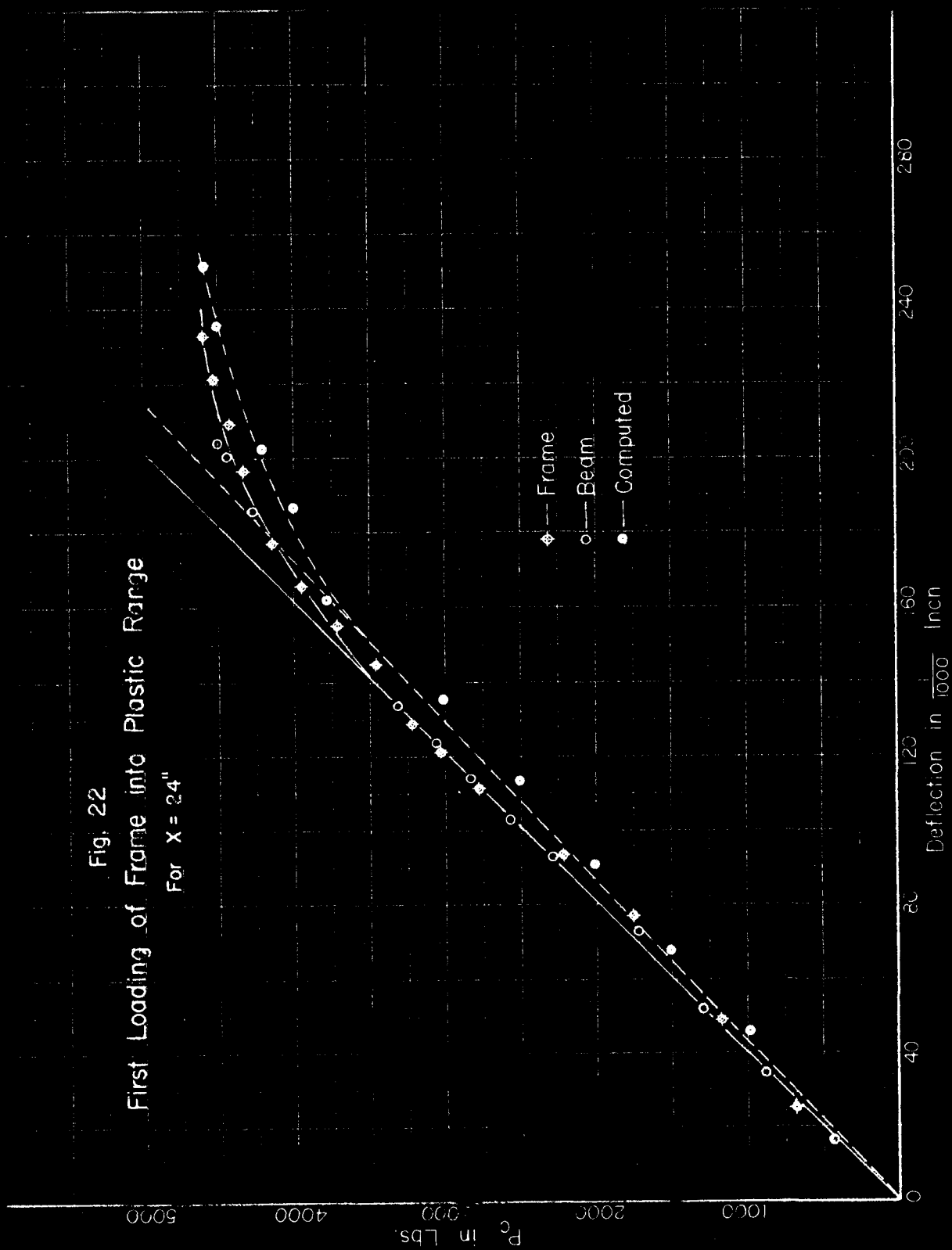
Plate 2

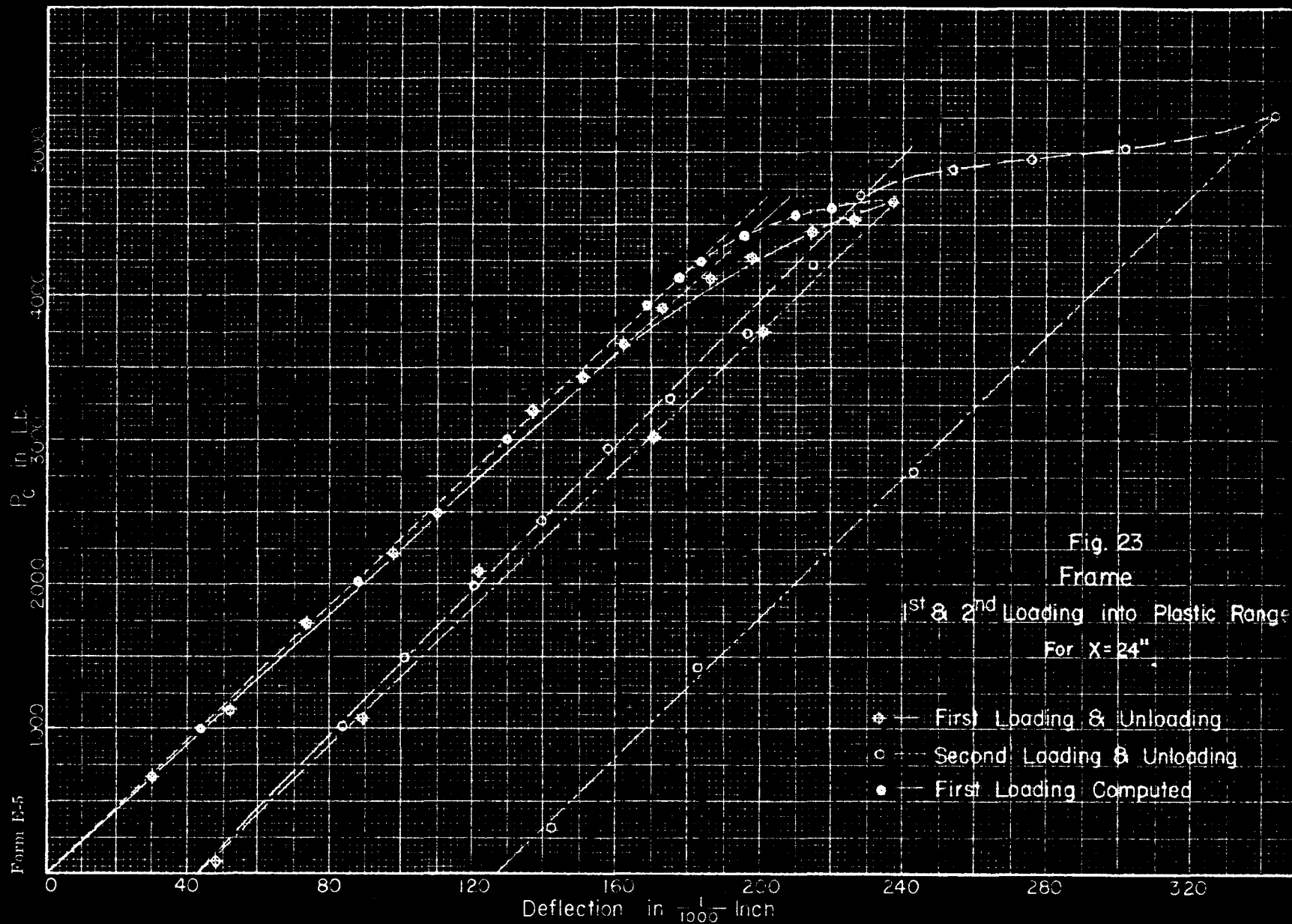


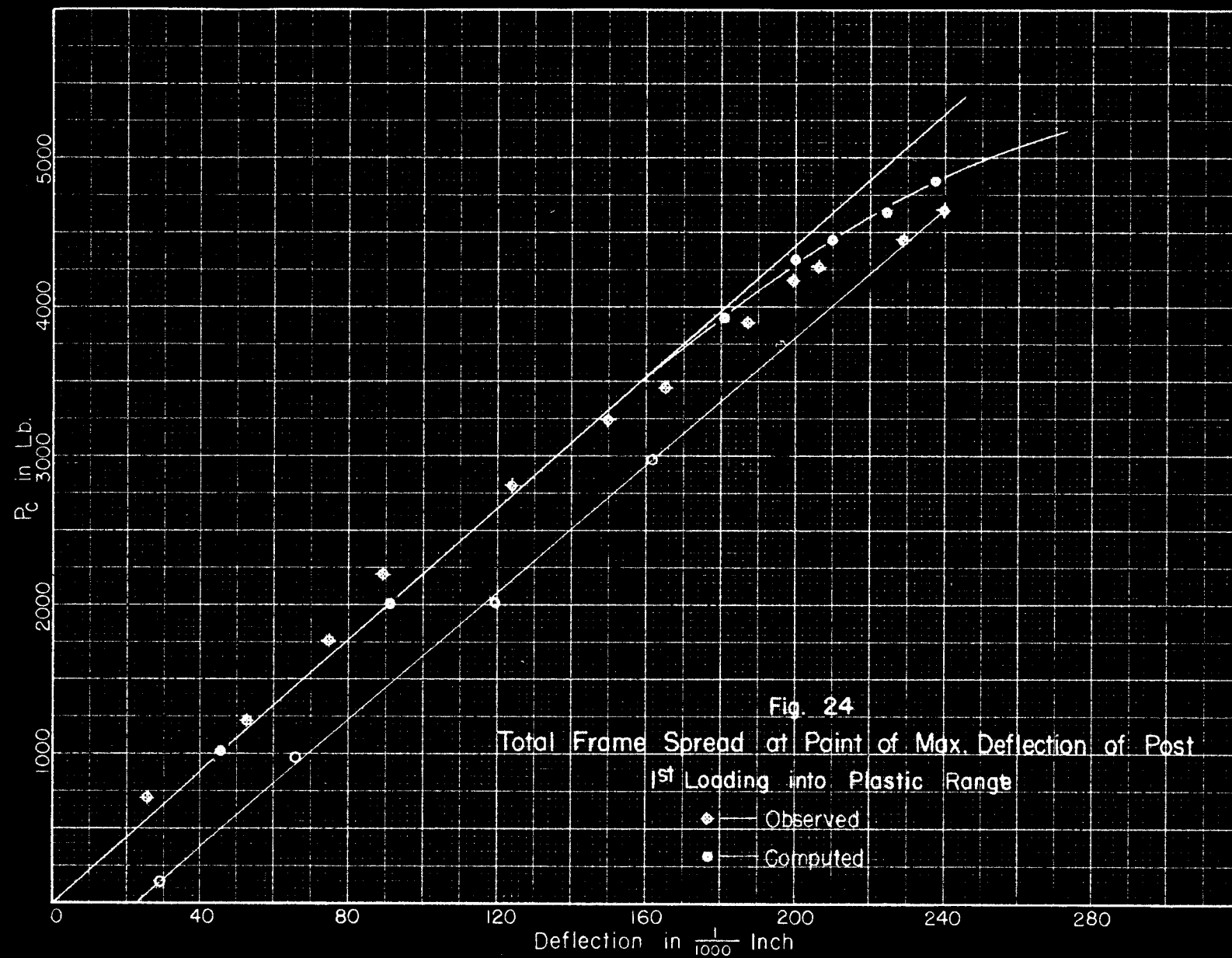
IV. RESULTS

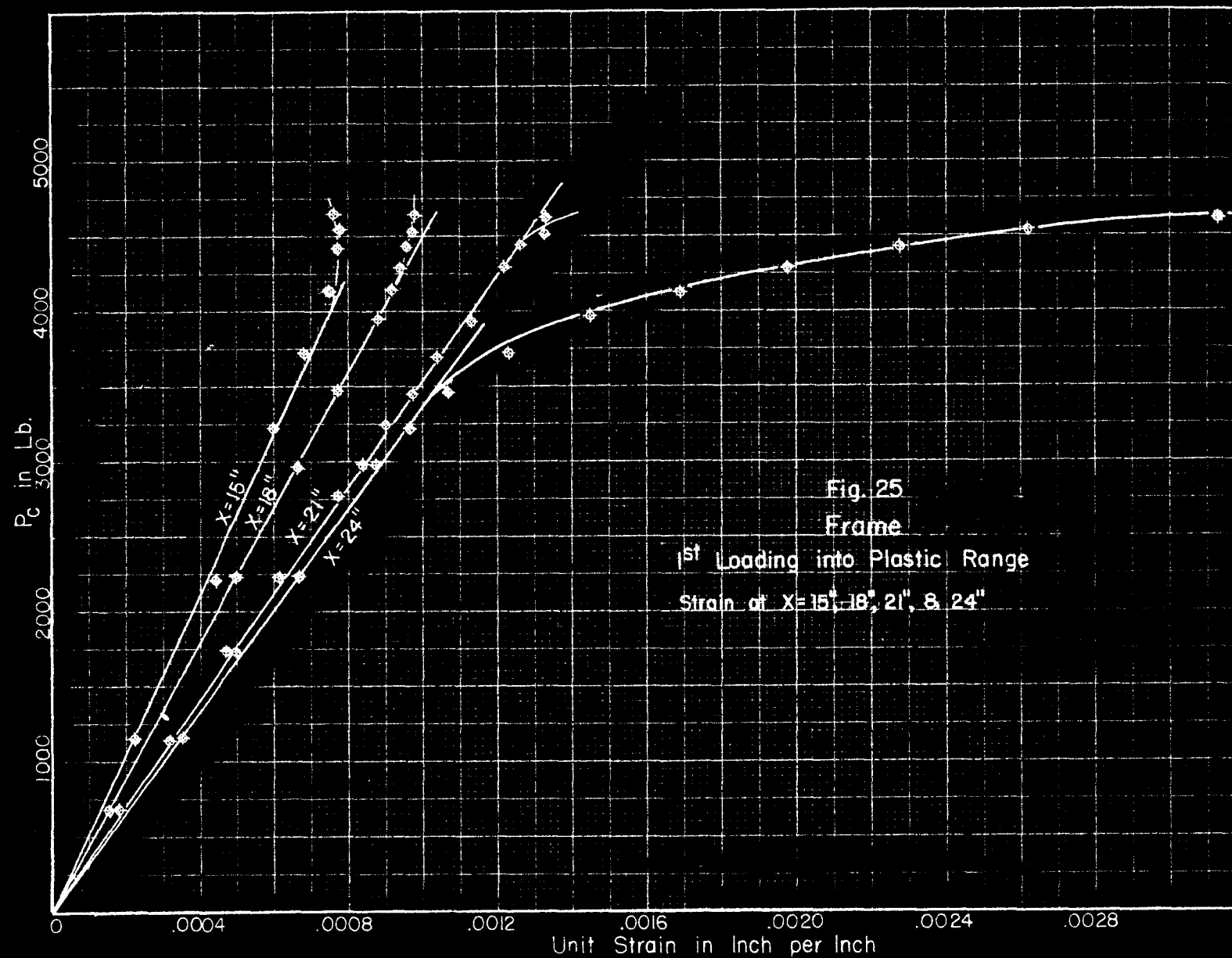


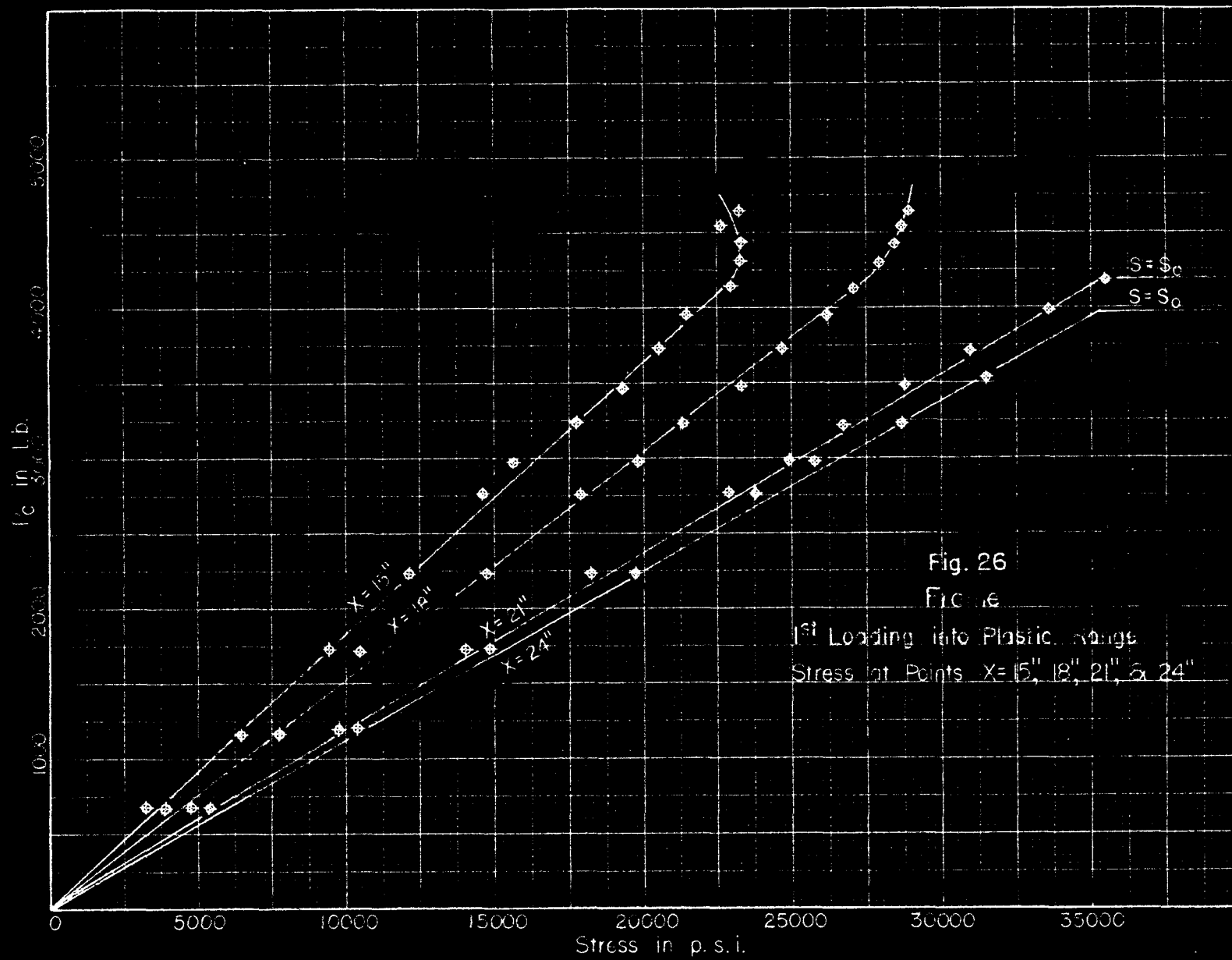












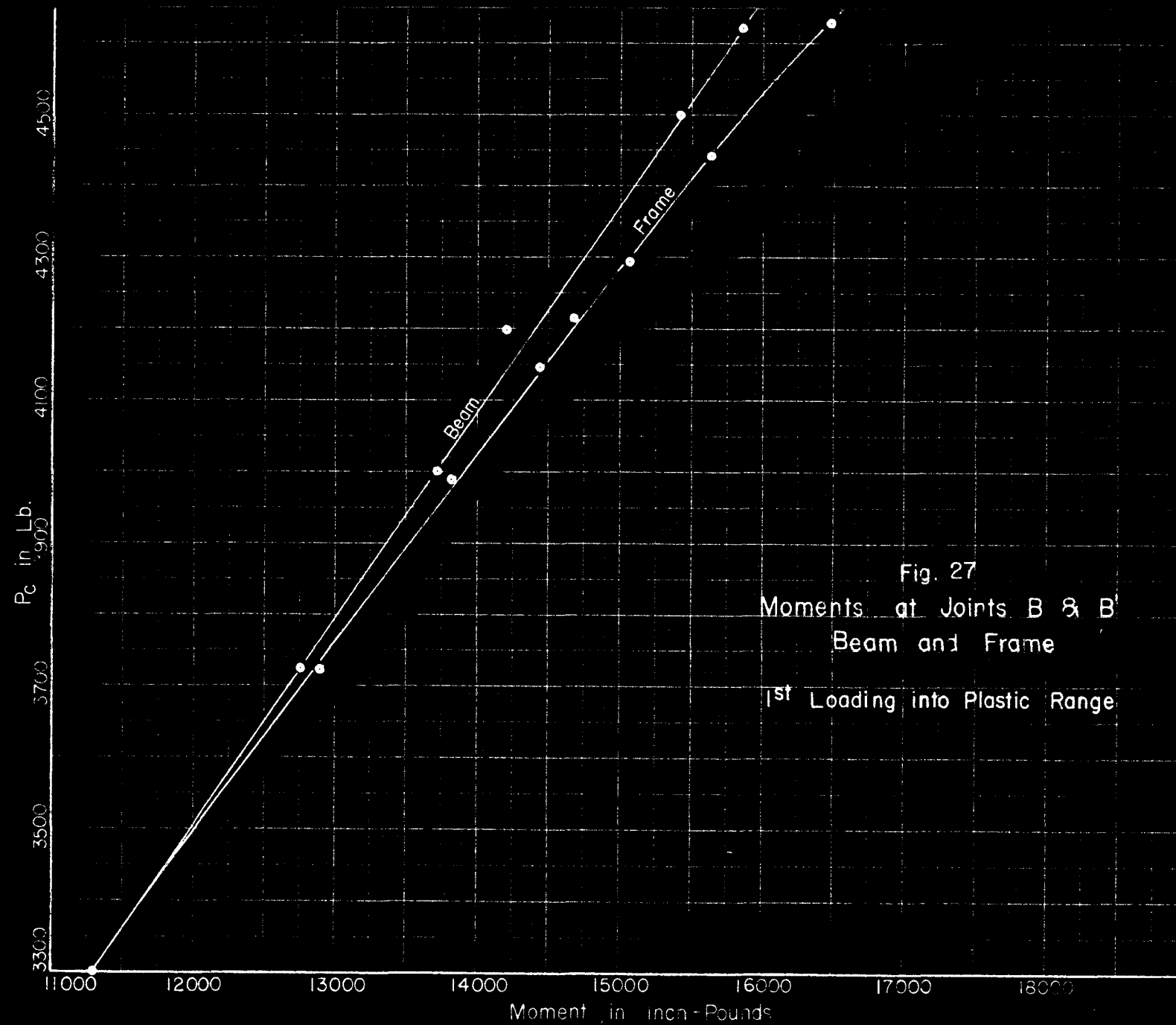
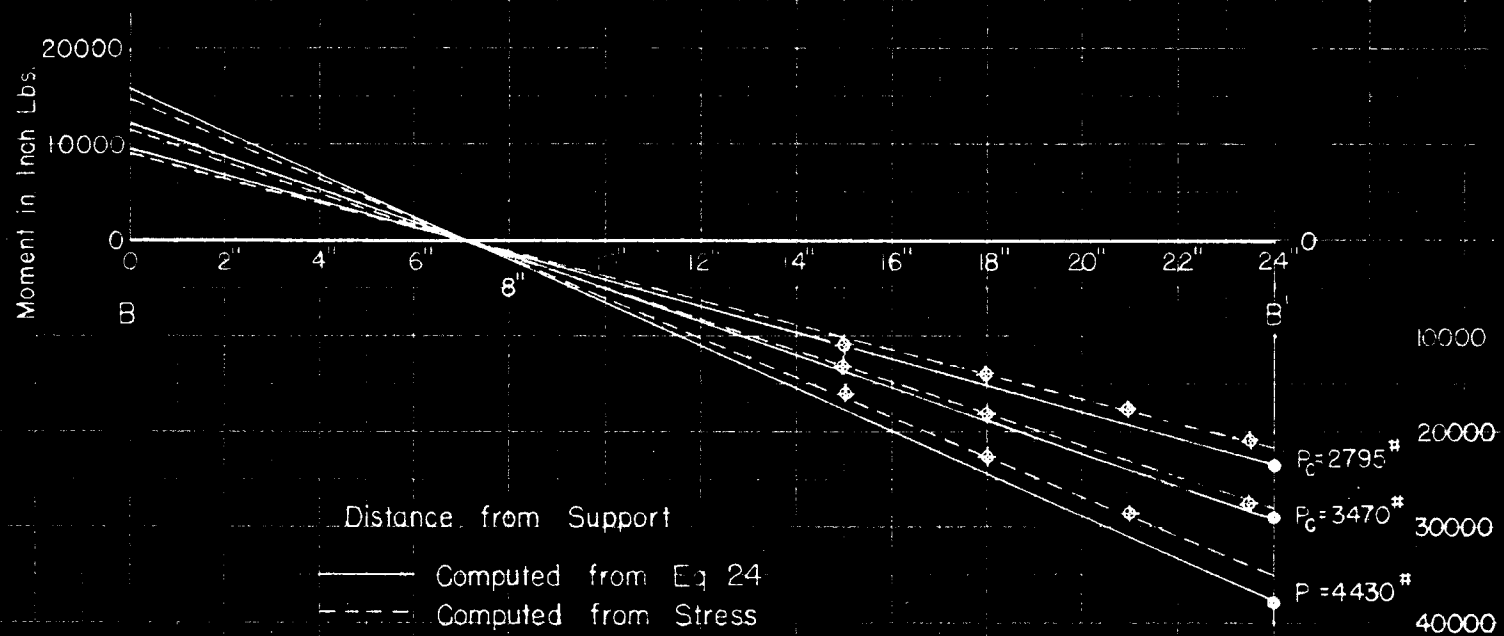


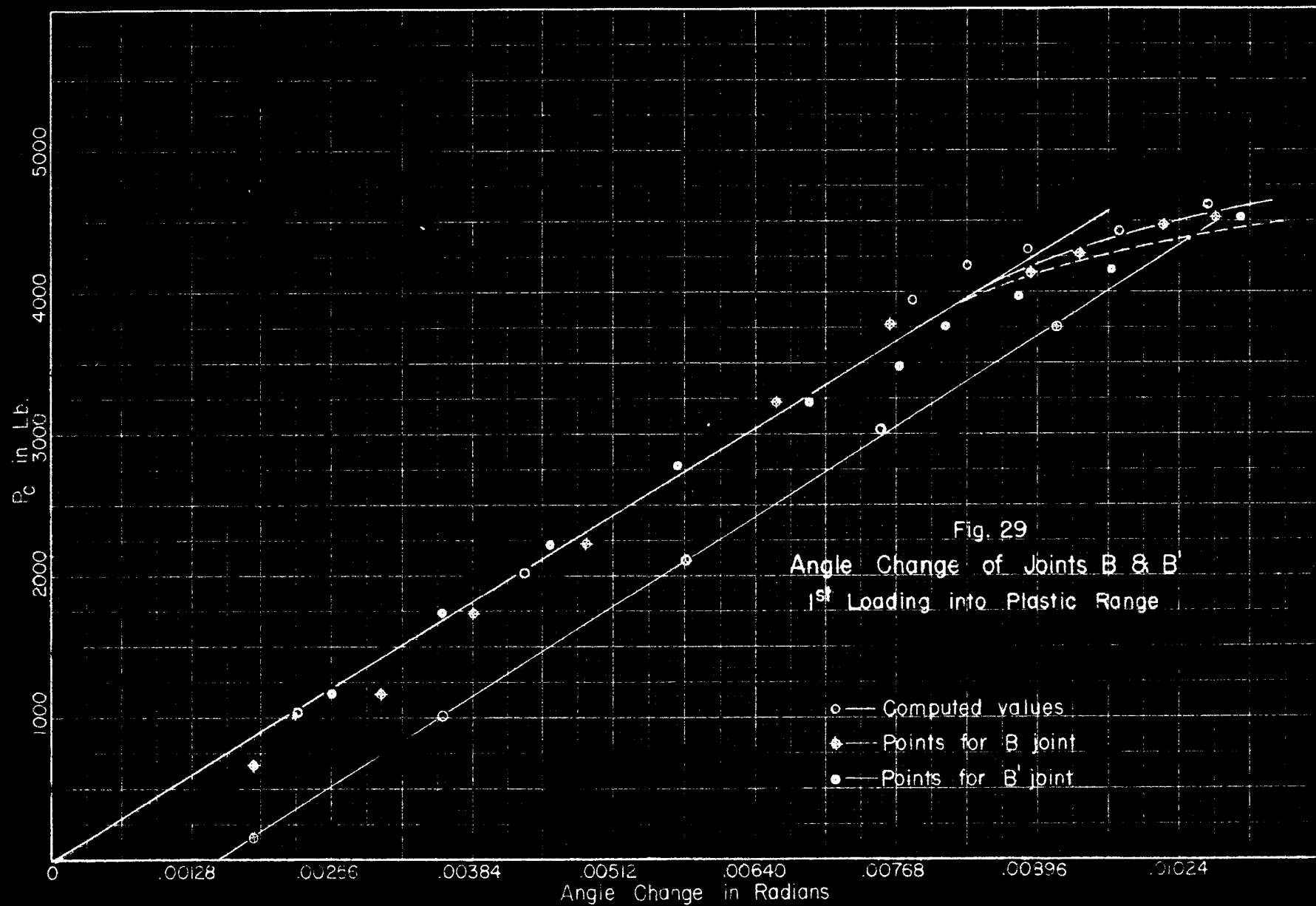
Fig. 27
Moments at Joints B & B'
Beam and Frame
1st Loading into Plastic Range



Moment Curve Along Top Member (B-B')

1st. Loading into Plastic Range

Fig. 28



V. DISCUSSION OF RESULTS

A. Beam

Figure 20 shows the deflection of the center point of the beam. The computed values are shown along with the observed values. The unloading curve for the beam is also shown in this figure.

The theoretical values, represented by the curve in Figure 20, when the beam is loaded below the yield point of the material is computed from the well known equation found by elementary theory. The observed curve does not coincide with the computed load-deflection curve when the beam is acting elastically. It is to be noted that the computed values for this curve are found by solving equation 10, and the value of E used in the equation is an experimentally determined value of E from tensile tests and a flexure test. It is not unlikely, then, if all other quantities are measured correctly the E value alone accounts for the two curves not coinciding. This indicates the importance of E being determined correctly for use in the equation. If the value of this 'apparent E ' for the beam is desired it may be found by computing the ratio of the ordinate of the computed curve to observed curve, and multiplying this by the experimentally determined E .

In the plastic range it is seen from Figures 20 and 21 that the computed values are higher than the observed values. This is expected since the curves do not coincide for an elastic loading. It must be added that

the difference between computed and observed may possibly be due to E , and the yield point stress, s_0 . In the plastic range the values of deflection are found by solving equations 18 and 20.

The value of yield point stress used in the computations for this investigation was determined by a tensile test of a specimen from the same material as the beam and the value is 35,600 pounds per square inch. In the test of the beam it was not possible to determine the value of the yield point load exactly because of the testing method used. It, however, could be determined as closely as ± 190 pounds. This would only affect the yield point stress in this investigation by an amount of 5 per cent of the stress. For this investigation it was felt that the predetermined value of the yield point stress should be used, because of the more accurate method of making the tensile and flexure test (Figures 34 and 35).

The unloading curves drawn from the data show a residual deflection for all the points at which the dials were placed. This is as it should be since there is permanent strain in the middle part of the beam. The unloading curves indicate a slope approximately equal to the slope of the loading curve in the elastic range. This, also, is as would be expected, since the unloading curve for the tensile test specimen is also a straight line, and is approximately equal in slope to the loading curve.

The second load curve, Figure 20, shows a rise in the yield point load which is expected. This "ageing" effect is shown by mild steel when it has been strained beyond the yield strain. It is known from other experimental tests that the amount of rise depends upon the initial permanent strain and also upon the length of time between the application

of the load. The slope of this second loading curve, Figure 22, is approximately the slope of the first load-deflection curve.

Since the slope of the second loading curve for a tensile test is approximately equal to the slope for the first loading in a tensile, the second deflection curve should approximate the first load-deflection curve.

B. Frame

The frame deflection values are shown in Figures 21, 22, and 23. The frame deflections in Figure 21 are very nearly equal to the beam deflections when both are acting elastically. This indicates that the effect of the horizontal force in the beam of the frame is negligible. Further, the spread of the frame posts does not affect the deflection values. Since the beam was tested with end moments equal to those of the frame it was naturally expected that the deflection values of the two beams would be nearly the same. The deflections are not in close agreement when the frame is acting plastically. Figures 21 and 22 show this. The reason for the deflection values not agreeing is found in the end moment values. The end moments of the frame are determined from equation 25 and for the beam they are determined according to the theory of elasticity. The values are shown in Figure 27.

The computed values of deflection in Figures 20 and 21 are determined from equation 10 and using the end moment value determined for the beam.

Figure 23 indicates that the loading and unloading curves for the frame are very similar to the beam curves in the figure. Figure 23 shows

approximately the same rise in the yield point as observed in the beam.

The computed values of the deflections for the frame are less than those observed. Here again, as in the beam, it is important that the value of E and s_0 be known accurately. We expect these values to be slightly less than the observed values since the shear deflection and the deflection due to the horizontal force upon the beam member of the frame is neglected. The end moments in this formula for deflection are determined by an approximation method, see solution of equation 25. If these moment values are slightly in error this error is introduced into the deflection equation in the same amount. Of course the accuracy here may be increased by making use of a computing machine. It is seen also from formula 25, that the moment itself is dependent upon the accuracy of E and s_0 .

Figure 24 shows the results of the data on frame spread. The frame spread values check very closely the values computed from equation 22. This is additional evidence of the accuracy of the moment values computed for the frame by equation 25.

Figure 25 shows the results of the strain measurements. These strains were measured at four points for all loads upon the frame. The strains were converted to stresses, Figure 26, and these values used to compute the moment at the point. Figure 28 shows the computed moment values from the strains and the theoretical values from equation 25. This figure (25) indicates that the strains at no point along the beam are linearly proportional to the load after the yield point is reached. Further evidence is found by studying Figure 26. This is expected because the inflection point in the beam is changed once the yield point load is passed.

Further investigation of the change of the inflection point will reveal the fact that the corner moments increase proportionally more rapidly than the center moment of the beam. This indicates that the elastic part of the beam will assume more of the moment once a part of the beam is strained beyond the yield strain. In this investigation one notes that the M_0/M_E ratio is decreased from 2.5 to approximately 2.30.

Figure 29 shows the results of the angle-change data. The values computed check very closely the values measured. The values computed were determined from equation 24 and this close agreement between the values proves that the joints are acting as assumed. The welded joint was assumed to be acting elastically and the material at the joint was assumed to be the same as the rest of the beam and post.

From this investigation it is clear that since the measurements of small quantities corresponding to static loads are necessary it is important that one should be very cautious in the extension and interpretations of the experimental results. Extreme care was exercised in this investigation to eliminate the experimental errors that may be present in an investigation of this kind. The close agreement of the measured and experimental results indicates that no error of consequence was made in the tests. Too, these tests support the assumptions made in the theory developed.

It should be noted that any interpretations of the data would not be complete unless consideration is given to two values that enter into the equations for predicting deflections of beam or posts, and angle changes. It is important to know their effects and the effects of these two quantities in the interpretations and the extension of the results of the

investigation. The two values are, namely, the yield point stress and the modulus of elasticity.

The equations herein developed require that the yield point stress be reasonably accurate if its value is assumed for a paper analysis of a frame or a beam. If it is determined experimentally, from the results obtained in this investigation, the value should be obtained in two ways - the standard tensile tests and flexure tests. The results are shown by Figures 34 and 35.

If equation 4 is examined it will be seen that a large error in the yield point value will result in a corresponding error in the value of the internal resisting moment. This is not the only effect of an incorrect value of the yield point stress. It is evident that the constant is dependent upon the yield point stress and any error in the stress is proportionally reflected in the value of the constant. Further, if the constant 'A' is in error the value of C_1/EI will likewise be in error. If C_1/EI is in error it affects the deflection values to a considerable degree. For instance, at the center the error is approximately the error in the angle change multiplied by one half of the length of the beam.

The modulus of elasticity should be determined as accurately as the yield point stress when possible. This quantity also affects the value of the constant "A." As mentioned above the quantity "A" affects the values to a great degree and large errors in its value are not permissible.

Further results of the second loading for the beam and the frame are given in the appendix and no conclusions are drawn from them. It may be seen that the load-deflection curve of the beam and the frame are very

nearly coincidental. Further, all the curves shown are straight up to the yield point load which is higher than the first yield point load. The curves appear straight beyond this new yield point load.

VI. SUMMARY

The investigation presents a method of analysis of a rigid frame built of mild steel, when the horizontal beam member is stressed beyond the yield point of the material. The frame is a 'U' type made up of rectangular members with welded joints. Prediction equations have been developed for deflections, frame spread, angle changes of the joints, moments, and shears. Deflections, frame spread, angle changes, and moments have been measured and compared with predicted values.

A restrained beam of mild steel, rectangular in cross section, was tested with a concentrated load at the center and equal moments on the ends, and the deflections measured. Prediction equations of deflection have been developed and compared with measured values of deflection.

Testing method and procedure have been developed for a rigid frame and a restrained beam, involving ductile behavior, for use in the laboratory.

VII. CONCLUSIONS

The experimental investigation and the mathematical analysis in this study furnished sufficient data to justify the following conclusions:

(1) The deflection formula developed is satisfactory and adequate for the determination of such values in a 'U' type frame, fixed at the ends of the posts and made up of members of rectangular cross section. The material is assumed to be isotropic.

(2) The investigation shows the adequacy of the formula herein developed for the angle changes at the joints of a 'U' type of rigid frame.

(3) The mathematical analysis shows a satisfactory method for the determination of moments and shears of a rigid frame, 'U' type, when plastic action exists in the beam member of the frame.

(4) The deflection formulas developed are applicable to a restrained beam of rectangular or square cross section.

(5) The investigation method and analysis are applicable to any 'U' type of frame, made up of members of rectangular or square cross section and isotropic material, symmetrically loaded.

VIII. BIBLIOGRAPHY

1. Bach, C. Elastizität und Festigkeit. 6th ed. pp. 258-265. Berlin. Julius Springer. 1911.
2. Bleich, F. Stahlhochbauten. Vol. 1, pp. 396-411. Berlin, Julius Springer. 1932.
3. Love, A. E. H. Treatise on the Theory of Elasticity. Vol. 1, p. 2, Cambridge, The University Press. 1892.
4. Meyer, E. von. Die Berechnung der Durchbiegung von Stäben, deren Material dem Hookeschen Gesetz nicht folgt. Zeitschrift des Vereines deutscher Ingenieure. Vol. 58, pp. 167-173. 1908.
5. Moore, H. F. History of the Flexure Formula. Proceedings of the Society for the Promotion of Engineering Education. Vol. 21, pp. 156-163. 1930.
6. Nadai, Árpád. Der bildsame Zustand der Werkstoffe. Chap. 22-23. Berlin, Julius Springer. 1927.
See also English Translation. Plasticity; a Mechanics of the Plastic State of Matter. Revised and enlarged from the first German edition. New York, McGraw-Hill. 1931.
7. Pinegin, W. von. Versuche über den Zusammenhang von Biegezugfestigkeit bei Gusseisen. Zeitschrift des Vereines deutscher Ingenieure. Vol. 56, pp. 2029-30. 1906.
8. Saint-Venant, B. de. Complément aux mémoires du 7 mars 1870 de M. de Saint-Venant et du 19 juin 1870 de M. Levy sur les équations différentielles 'indéfinies' du mouvement intérieur des solides ductiles etc.; - Équations définies ou relatives aux limites de ces corps; - Applications. Journal de Mathématique. Vol. 16, pp. 380-382. 1870.
9. Scott, E. O. Deformations of Beams Involving Ductile Behavior. Unpublished Ph. D. Thesis. University of Michigan. 1939.
10. Van Den Broek, J. A. Limit Design. Proceedings American Society of Civil Engineers. Vol. 66, No. 8, part 2. 1940.
11. Winter, George. Discussion of Limit Design. Proceedings American Society of Civil Engineers. Vol. 66, No. 8, part 2. 1940.

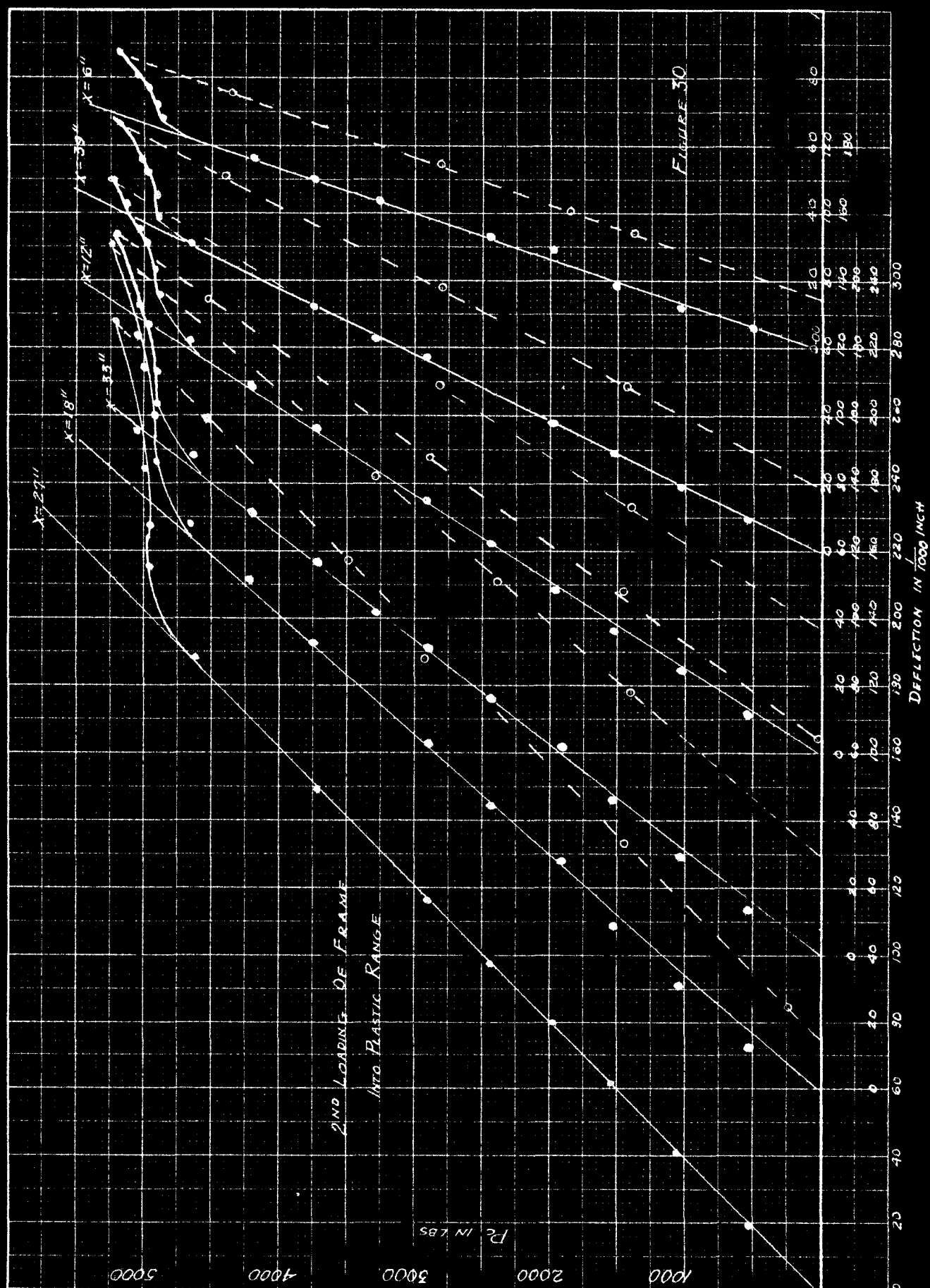
IX. ACKNOWLEDGMENTS

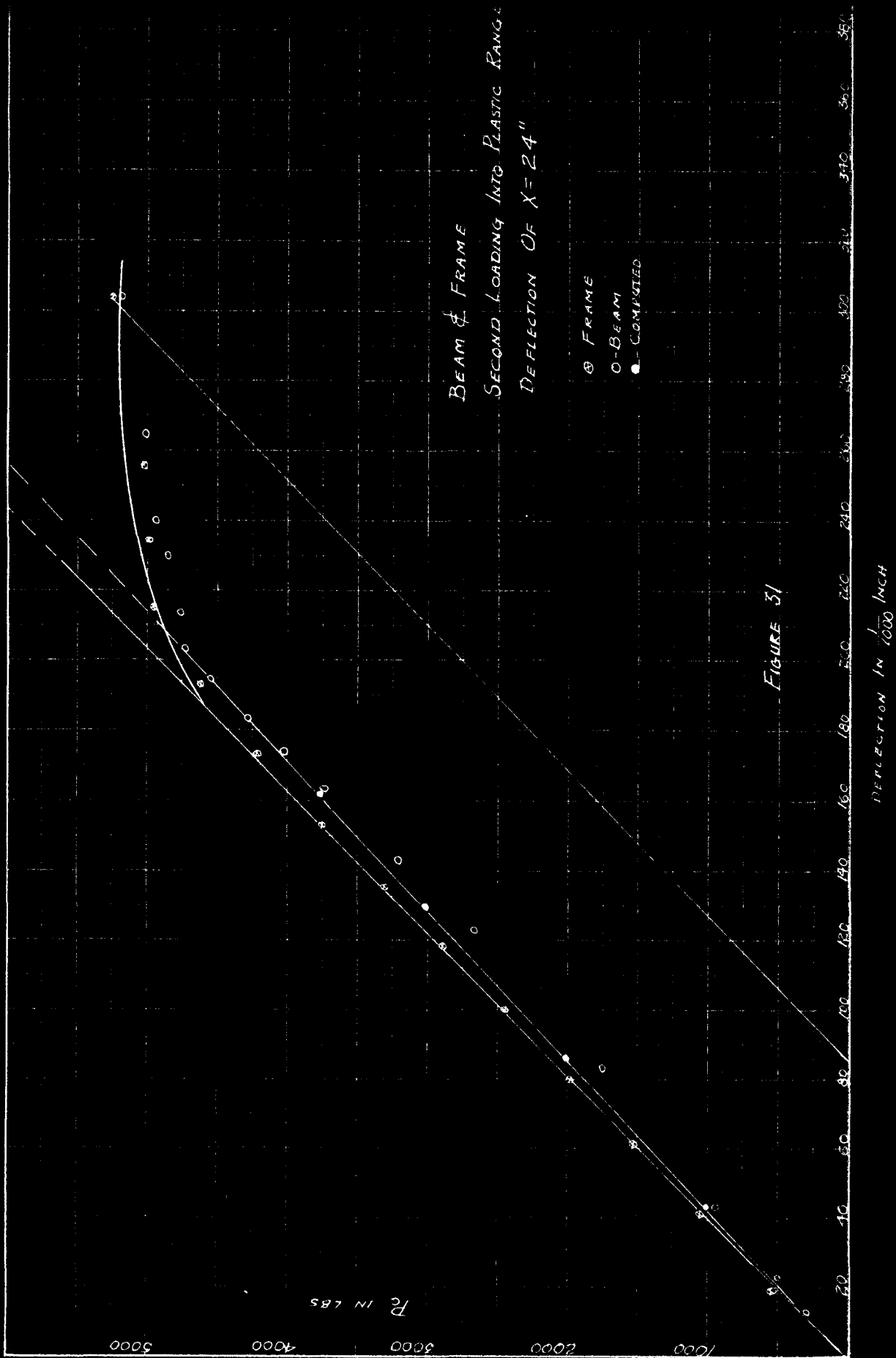
The writer wishes to express his gratitude to Dr. Glenn Murphy, Professor of Theoretical and Applied Mechanics, for his direction of the investigation to its successful completion. To Mr. R. A. Caughey, Professor of Civil Engineering, and co-chairman of the writer's graduate committee, the writer is also grateful for his valuable suggestions and direction in the experimental phase of the investigation. To Professor H. J. Gilkey, Head of the Department of Theoretical and Applied Mechanics, and Professor L. O. Stewart, Head of the Department of Civil Engineering, the writer is grateful for their assistance in obtaining the materials and workmanship necessary for the experimental part of the investigation. To Mr. Richard Othmer, Graduate Assistant, and Mr. W. E. Paley, the writer is grateful for the aid they gave in the measurements of the quantities in the experimental work.

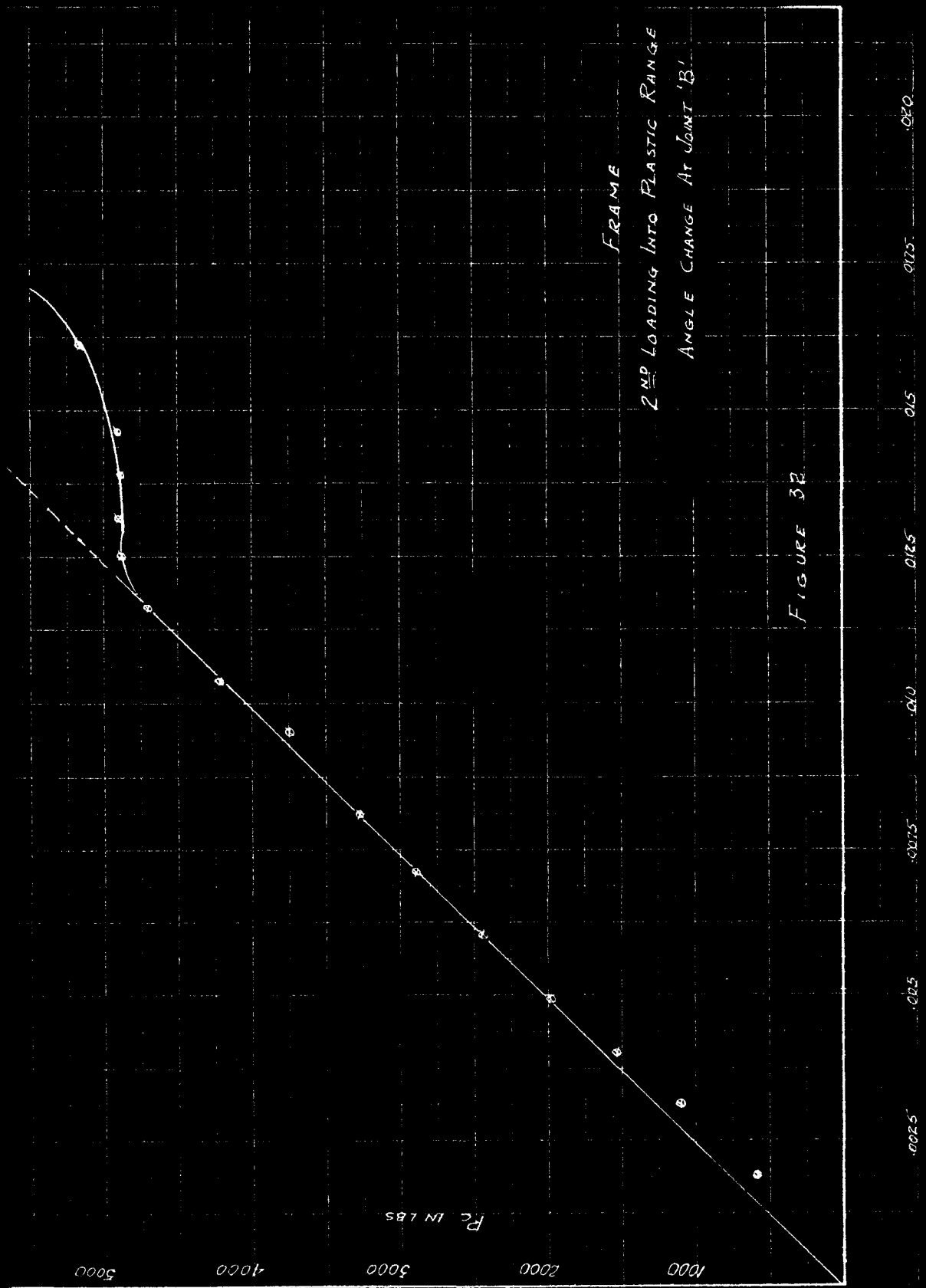
X. VITA

Walter Thomas Daniels, the son of Thomas Charles Daniels and Bertha Long Daniels, was born in Fort Clark, Texas, April the 26th, 1908. He attended elementary school at Brackettville, Texas, and high school at Wichita, Kansas, and Columbus, New Mexico, graduating from high school at Columbus, New Mexico, June 6, 1923. He attended Prairie View College, Prairie View Texas, from 1923 to 1924, and following this enrolled at Arizona University in 1924-25, and 1926-29, graduating June 5th, 1929, with the degree Bachelor of Science in Civil Engineering. From 1929 to 1931 he was employed as Assistant Professor of Mechanic Arts at Prairie View College, Prairie View, Texas. In September 1931 he enrolled in the graduate college at The Iowa State College and graduated December 22, 1932, with the degree Master of Science in Structural Engineering. Following this he worked with the T. A. Allen Construction Company, Nogales, Arizona, from August 1933 to September 1934. From September 1934 to September 1939 he was employed at A. & T. College, Greensboro, North Carolina, as Professor of Mechanic Arts. In September 1939 he enrolled at The Iowa State College for major graduate work in Theoretical and Applied Mechanics and Structural Engineering, and minor work in Physics. His research problem was conducted under the joint direction of Dr. Glenn Murphy, Professor of Theoretical and Applied Mechanics, and Mr. R. A. Caughey, Professor of Civil Engineering, at The Iowa State College.

XI. APPENDIX

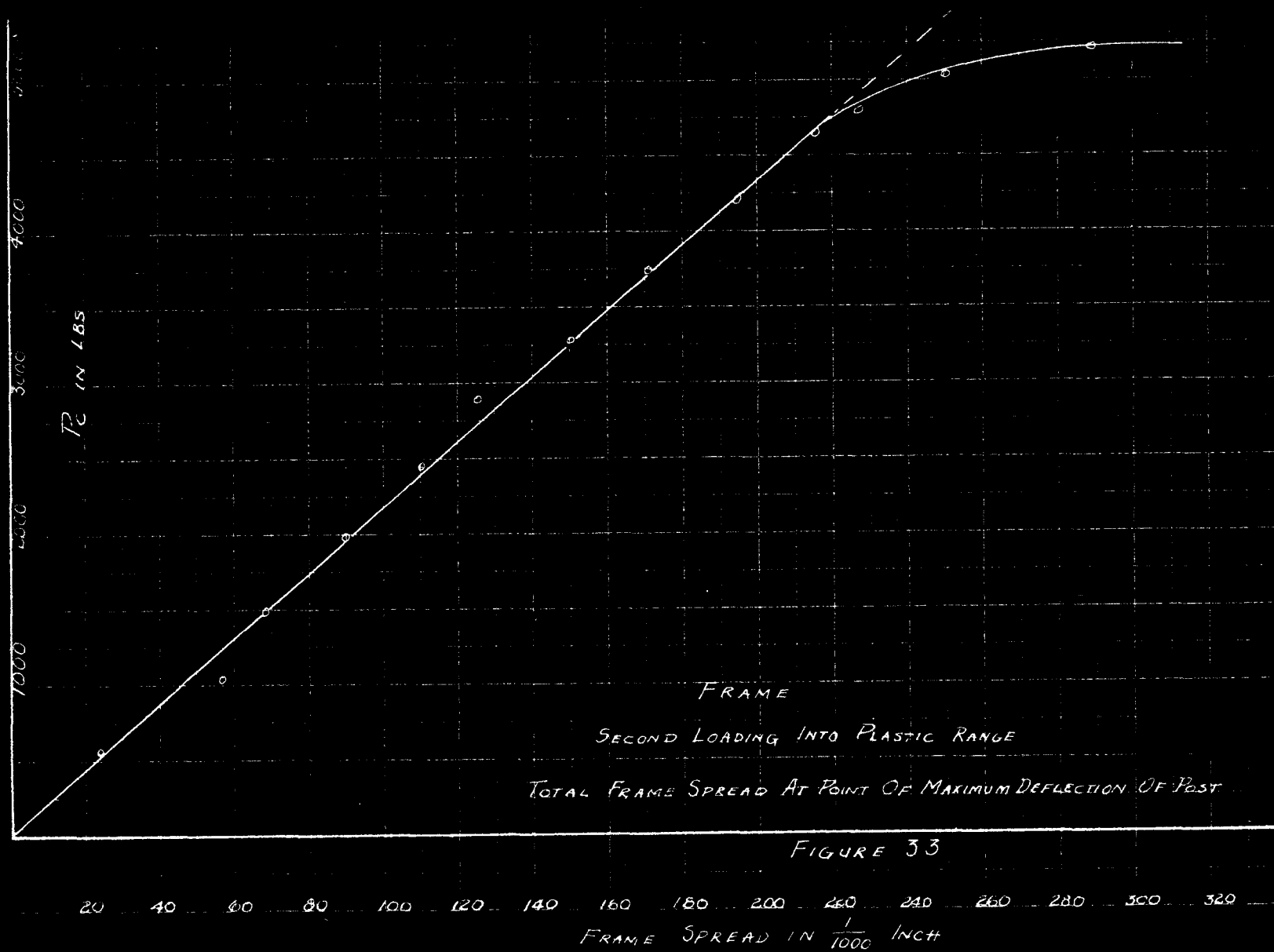






FRAME
 2ND LOADING INTO PLASTIC RANGE
 ANGLE CHANGE AT JOINT 'B'

FIGURE 32



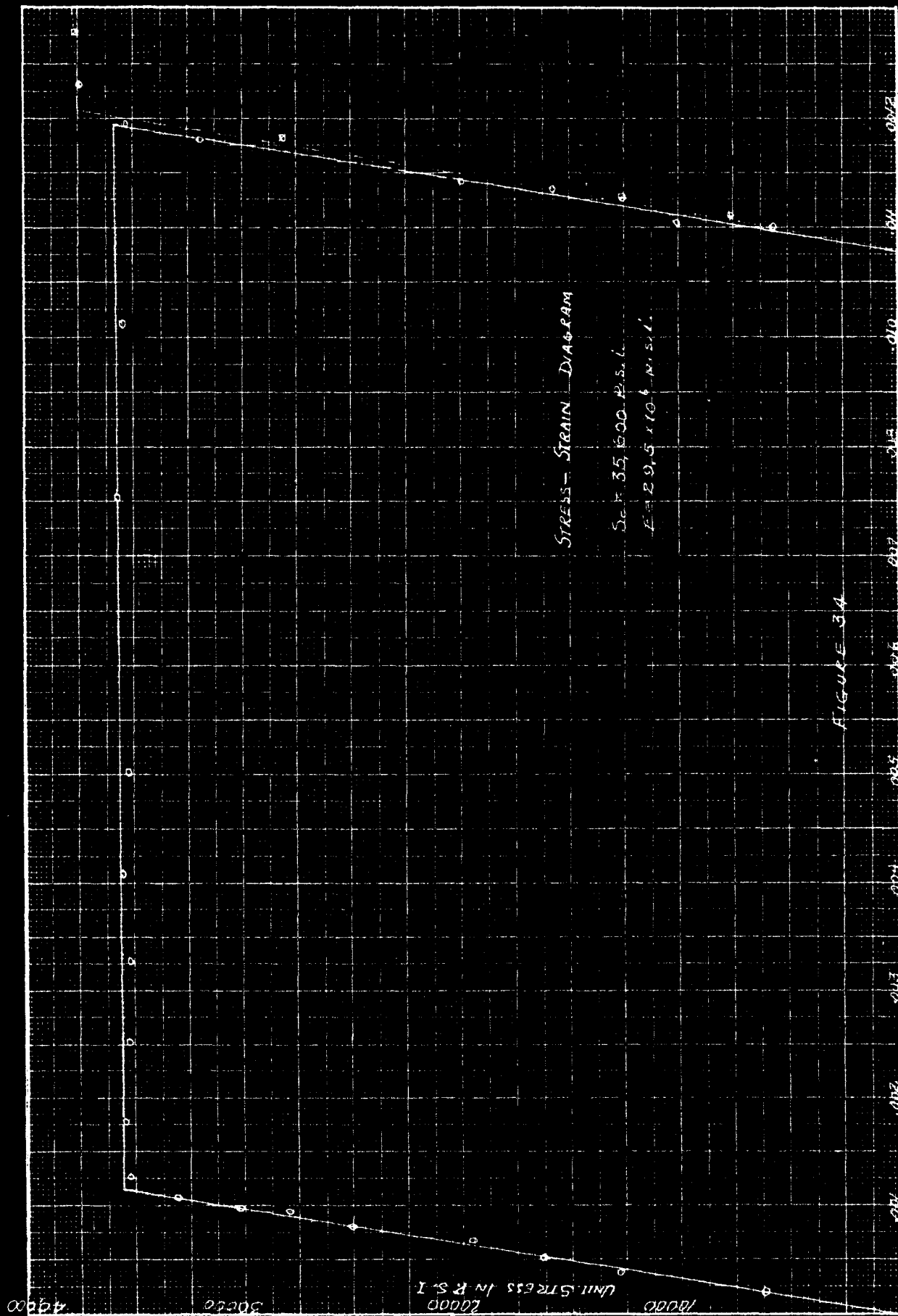


FIGURE 34

UNIT : PAIN IN INCH PER INCH

